Hecke Operators in K-theory of Bianchi Groups

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Abstract

This thesis is essentially an introduction to the study of Hecke operators acting on the equivariant K-theory of the classifying space for proper actions of Bianchi groups.

The document is divided in two parts. The first part gives definitions and generalities on proper actions, Bredon (co)homology, K-theory, and Hecke operators; these are given always thinking in the application to Bianchi groups, which are discrete groups of matrices. After this, we define a Hecke operator in K-theory using a decomposition by conjugacy classes of elements of finite order.

In the second part, we describe the algebraic structure of Bianchi groups, that is, their decomposition as amalgamated products, including the explicit decompositions for Euclidean Bianchi groups. We focus on the group $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z}[i])$, for which we compute group cohomology, Bredon cohomology, and equivariant K-theory of the classifying space for proper actions. Then, given a prime in $\mathbb{Z}[i]$, we define an associated congruence subgroup of Γ_1 in order to compute a Hecke operator in $K^*_{\Gamma_1}(\underline{E}\Gamma_1)$ factoring through the K-theory of this subgroup. We conclude with explicit calculations for $p = 1 + i$.

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Contents

1 Introduction

The study of Bianchi groups began, personally, as a natural next step from the study of the modular group $\mathrm{PSL}_2(\mathbb{Z})$. The theory of Hecke operators corresponding to modular forms and cohomology of congruence subgroups of the modular group has been widely studied, and also, has been generalized to other groups and different contexts.

The Bianchi group associated with a positive square-free integer d is defined as

$$\Gamma_d = \mathrm{PSL}_2(\mathbb{O}_d) = \mathrm{SL}_2(\mathbb{O}_d)\,/\,\{\pm\ \mathrm{Id}\}$$

where $\mathbb{O}_d$ is the ring of integers of the imaginary quadratic extension $\mathbb{Q}(\sqrt{-d})$. There are several interesting, important arithmetic properties of Bianchi groups, and they have gained a lot of interest in the last decades. For instance, one of the works that gave direction to this thesis is an article by Mesland and Şengün, where they study Hecke operators in the context of KK-theory and K-homology.

In this document, we study the classifying space for proper actions and the equivariant K-theory associated to Bianchi groups, specifically to the group Γ_1. A way to describe this space is considering the action of $\mathrm{PSL}_2(\mathbb{C})$ of the hyperbolic 3-space, which can be viewed as quaternions, then the classifying space can be seen as a subspace of it. This action also leads to some important properties of Bianchi groups, such as their decomposition as amalgamated products.

Then, following some of the geometrical interpretations of Hecke operators, we can define a Hecke operator in the groups $K^*_{\Gamma_d}(\underline{E}\Gamma_d)$. This is done by factoring the operator as a composition of restriction and corestriction, which means that we need to define morphisms going up and down between the K-theory associated to Γ_d and to a particular subgroup. All being well, these operators will have, or lead to, some arithmetic, algebraic, and/or geometric properties associated to the groups, as they already do in the context of modular forms and group cohomology.

Part I
K-theory and Hecke operators

We begin with an introduction to spectral sequences, where the Mayer-Vietoris spectral sequence is defined. Then, we define proper actions, Bredon (co)homology, K-theory and Hecke operators. The main core of this part is the interpretation of Hecke operators that leads to the corresponding definition in K-theory.

2 Spectral Sequences

Spectral sequences are algebraic objects used mostly in algebraic topology. We will describe briefly what this objects are and some of their applications. For a complete introduction and development of this subject see [20].

2.1 Basic notions

A **differential bigraded module** over a ring R, is a collection of R-modules, $\{E^{p,q}\}_{p,q\in\mathbb{Z}}$, together with an R-linear mapping $d : E^{*,*} \to E^{*,*}$, of bidegree $(s, 1-s)$, or $(-s, s-1)$, for some $s \in \mathbb{Z}$, such that $d \circ d = 0$. Sometimes we use the same indices in d to specify the particular domain. indexes The mapping d is called the **differential**. The bidegree (m, n) means that d goes from $E^{p,q}$ to $E^{p+m,q+n}$, for each pair p, q. With this, we can take the homology of a differential bigraded module with bidegree $(s, 1-s)$; we define it as

$$H^{p,q}(E^{*,*}, d) = \mathrm{Ker}(d : E^{p,q} \to E^{p+s,q+1-s}) \;/\; \mathrm{Im}(d : E^{p-s,q-1+s} \to E^{p,q}).$$

This gives another bigraded module $\{H^{p,q}(E^{*,*}, d)\}_{p,q\in\mathbb{Z}}$. The same can be done with a differential bigraded module of bidegree $(-s, s-1)$.

Now we can give the definition.

Definition 2.1. *A **spectral sequence** is a collection of differential bigraded R-modules $\{E_r^{*,*}\}_{r=1}^{\infty}$, such that the differentials are either all of bidegree $(-r, r-1)$ (for homological type) or all of bidegree $(r, 1-r)$ (for cohomological type) and, for all p, q, r, there is an isomorphism $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$. The module $E_r^{*,*}$ is called the E_r**-term** of the spectral sequence.*

It is important to mention that $E_r^{*,*}$ and d_r determine the $E_{r+1}^{p,q}$, but not d_{r+1}, so one term of the spectral sequence is not enough to describe it all.

There is another way to describe an spectral sequence. Let $\{E_0^{p,q}\}_{p,q\in\mathbb{Z}}$ be a family of R-modules. Suppose that for each $p, q \in \mathbb{Z}$ there is a tower of submodules

$$B_0^{p,q} \subset B_1^{p,q} \subset \cdots \subset B_n^{p,q} \subset \cdots \subset Z_n^{p,q} \subset \cdots \subset Z_1^{p,q} \subset Z_0^{p,q} \subset E_0^{p,q}$$

together with short exact sequences

$$0 \longrightarrow Z_{n+1}^{p,q}/B_n^{p,q} \longrightarrow Z_n^{p,q}/B_n^{p,q} \longrightarrow B_{n+1}^{p+n+1,q-n}/B_n^{p+n+1,q-n} \longrightarrow 0.$$

These define a spectral sequence by setting $E_{n+1}^{p,q} = Z_n^{p,q}/B_n^{p,q}$ and

$$d_n^{p,q} : E_n^{p,q} = Z_{n-1}^{p,q}/B_{n-1}^{p,q} \longrightarrow B_n^{p+n,q+1-n}/B_{n-1}^{p+n,q+1-n} \subset E_n^{p+n,q+1-n},$$

taken from the exact sequences. See [20] for the complete explanation.

A spectral sequence is said to **collapse at the N-th term** if $d_r = 0$ for $r \geq N$. This would imply that $E_r^{*,*} = E_{r+1}^{*,*}$, then we define the **limit term**, E_∞, as $E_N^{*,*}$.

Let F^* be a **filtration** on an R-module M, that is, a family of submodules $\{F^pM\}_{p\in\mathbb{Z}}$, which could be decreasing, so $F^{p+1}M \subset F^pM$, or increasing, so $F^pM \subset F^{p+1}M$, such that

$$\bigcap_{p\in\mathbb{Z}} F^pM = 0 \qquad \text{and} \qquad \bigcup_{p\in\mathbb{Z}} F^pM = M.$$

Now, define its **associated graded module**, $E_0^*(M)$, as

$$E_0^p(M, F) = \begin{cases} F^pM/F^{p+1}M, & \text{for } F^* \text{ decreasing;} \\ F^pM/F^{p-1}M, & \text{for } F^* \text{ increasing.} \end{cases}$$

Also, if we have a graded R-module M^*, in order to examine its filtration on each degree, we define $F^pM^k = F^pM^* \cap M^k$ and the associated graded module as

$$E_0^{p,q}(M^*, F) = \begin{cases} F^pM^{p+q}/F^{p+1}M^{p+q}, & \text{for } F^* \text{ decreasing;} \\ F^pM^{p+q}/F^{p-1}M^{p+q}, & \text{for } F^* \text{ increasing.} \end{cases}$$

Using these notions, we say that a spectral sequence $\{E_r^{*,*}, d_r\}_r$ **converges** to a graded R-module M^* if there is a filtration F^* on M^* such that, for each p, q,

$$E_\infty^{p,q} \cong E_0^{p,q}(M^*, F).$$

If M has a differential structure, then we can construct a spectral sequence from it. We call an R-module M a **filtered differential graded module** if

- M is a direct sum of submodules, $M = \bigoplus_{n=0}^{\infty} M^n$;
- there is an R-linear mapping, $d : M \to M$, of degree 1 (so $d : M^n \to M^{n+1}$) or degree -1 (so $d : M^n \to M^{n-1}$) satisfying $d \circ d = 0$; and
- M has a filtration F^* and the differential d respects the filtration, which means that $d : F^pM \to F^pM$.

Then, we have the next theorem.

Theorem 2.2. *Each filtered differential graded module (M, d, F^*) determines a spectral sequence, $\{E_r^{*,*}, d_r\}_{r=1}^{\infty}$, with d_r of bidegree $(r, 1-r)$ and*

$$E_1^{p,q} \cong H^{p+q}(F^pM/F^{p+1}M).$$

If we suppose further that the filtration is bounded, that is, for each dimension n, there are values $s = s(n)$ and $t = t(n)$, so that

$$0 = F^sM^n \subset F^{s-1}M^n \subset \cdots \subset F^{t+1}M^n \subset F^tM^n = M^n,$$

then the spectral sequence converges to $H(M, d)$, that is,

$$E_\infty^{p,q} \cong F^pH^{p+q}(M, d)/F^{p+1}H^{p+q}(M, d).$$

For a proof, see [20].

In general, there are much more different (weaker, stronger) algebraic structures on which spectral sequences can be used, such as taking R to be a graded ring, or M to be a graded vector space, a graded algebra, or simply a graded group (a direct sum of groups).

Spectral sequences are objects that usually are given with the purpose of discovering or describing some graded object M, although arriving to M may be impossible without enough information. One of the main obstacles could be the so-called *extension problem*. We will discuss this in the next paragraphs.

Suppose that there is a spectral sequence which converges to a graded object M and that we already know what is the limit term E_∞, but without any relation between the $E_\infty^{p,q}$. The convergence guarantees that there is a (decreasing) filtration F^* on M for which, for all p, q,

$$E_\infty^{p,q} \cong F^p M^{p+q} / F^{p+1} M^{p+q}.$$

This is the same as saying that there are short exact sequences

$$0 \longrightarrow F^{p+1} M^{p+q} \longrightarrow F^p M^{p+q} \longrightarrow E_\infty^{p,q} \longrightarrow 0 \tag{1}$$

for any p, q.

We cannot say much more without assuming something else.

For the case when M is a graded finite vector space, it can be recovered taking the direct sum, for any n,

$$M^n \cong \bigoplus_{p \in \mathbb{Z}} F^p M^n / F^{p+1} M^n \cong \bigoplus_{p+q=n} E_\infty^{p,q},$$

since these vector spaces are determined up to isomorphism just with their dimension.

Now, going back to the general case, suppose that for some n_0, we have $E_\infty^{p,q} = 0$ whenever $p + q = n_0$, that is, the (anti-)diagonal is zero. Then, from the exact sequences (1) we can conclude that $F^{p+1} M^{n_0} = F^p M^{n_0}$ for all p, and then clearly $M^{n_0} = 0$.

Now, suppose that $E_\infty^{p,q}$ is not zero only for one pair p_0, q_0. We have

$$0 = \cdots = F^{p_0+2} M^{n_0} = F^{p_0+1} M^{n_0} \subset F^{p_0} M^{n_0} = F^{p_0-1} M^{n_0} = \cdots = M^{n_0}$$

but also we have an exact sequence

$$0 \longrightarrow F^{p_0+1} M^{n_0} \longrightarrow F^{p_0} M^{n_0} \longrightarrow E_\infty^{p_0,q_0} \longrightarrow 0,$$

so we obtain the isomorphism

$$M^{n_0} \cong E_\infty^{p_0,q_0}.$$

Following with the idea, suppose that $E_\infty^{p,q}$ is not zero only for two pairs p_0, q_0 and p_1, q_1, with $p_0 > p_1$. Then,

$$0 = \cdots = F^{p_0+1} M^{n_0} \subset F^{p_0} M^{n_0} = \cdots = F^{p_1+1} M^{n_0} \subset F^{p_1} M^{n_0} = \cdots = M^{n_0},$$

so the exact sequences

$$0 \to F^{p_1+1}M^{n_0} \to F^{p_1}M^{n_0} \to E_\infty^{p_1,q_1} \to 0, \quad 0 \to F^{p_0+1}M^{n_0} \to F^{p_0}M^{n_0} \to E_\infty^{p_0,q_0} \to 0$$

imply we have the exact sequence

$$0 \longrightarrow E_\infty^{p_0,q_0} \longrightarrow M^{n_0} \longrightarrow E_\infty^{p_1,q_1} \longrightarrow 0$$

and the isomorphism $E_\infty^{p_1,q_1} \cong M^{n_0}/E_\infty^{p_0,q_0}$. With this result, M^{n_0} is not determined yet, thus it depends on each particular case.

We see that, after this, with more than two non zero terms in a diagonal n, the object M^n is hard to describe.

In the next section we will discuss a spectral sequence that will be useful to compute the cohomology of the Bianchi group Γ_1.

2.2 The Mayer–Vietoris spectral sequence

The following spectral sequence obtains its name (although it is not standard) because it may be considered as the generalization of the Mayer–Vietoris long exact sequence relating the cohomology groups of two spaces and their union by a subspace. For a little more detailed construction and for alternative constructions see [26] and [29]. The spectral sequence is constructed in [29] for homology groups; here we will use the same method to obtain cohomology groups.

We will require a topological space that is a **simplicial complex**. These are the Δ-complexes whose simplices are uniquely determined by their vertices; this is the same as saying that each n-simplex has $n+1$ different vertices, and that no other n-simplex has this same set of vertices. Anyway, we will not use this concepts in the construction of the spectral sequence (but the assumption should be necessary for everything in the background to work well).

Let X be a simplicial complex and suppose there is a covering $\mathcal{U} = \{U_i\}_{i\in I}$ for X, where I is an ordered set. We define the **nerve** of $\mathcal{U}$ as the family $N(\mathcal{U})$ of finite subsets $\sigma \subset I$ for which the subspace $X_\sigma := \bigcap_{i\in\sigma} U_i$ is not empty. ($N(\mathcal{U})$ is obtained to be an abstract simplicial complex.)

For each $k \geq 0$, take $N_k(\mathcal{U})$ as the set of the $\sigma \in N(\mathcal{U})$ of order k and define the (co)chain complex

$$C^k = \bigoplus_{\sigma\in N_k(\mathcal{U})} C^*(X_\sigma)$$

where $C^*(-)$ denotes the cellular cochain complex (see [13] for a definition). Now, for each $0 \leq i \leq k$ there is a boundary map $\partial_i : N_k(\mathcal{U}) \to N_{k-1}(\mathcal{U})$, given by

$$\partial_i \sigma = \partial_i \{j_0 < \cdots < j_k\} = \{j_0 < \cdots < \widehat{j_i} < \cdots < j_k\},$$

which gives the inclusions $X_\sigma \hookrightarrow X_{\partial_i \sigma}$ that induce morphisms $\partial_i : C^*(X_{\partial_i \sigma}) \to C^*(X_\sigma)$. Then, taking $\partial = \sum_{i=0}^{k} (-1)^i \partial_i$ and extending linearly over the direct sum we obtain a chain map $\partial : C^{k-1} \to C^k$, $k \geq 1$.

With this, define

$$E_1^{p,q} = H_q(C^p) = \bigoplus_{\sigma \in N_p(\mathcal{U})} H^q(X_\sigma),$$

together with the differentials $d : E_1^{p,q} \to E_1^{p+1,q}$ induced from ∂. Here, $H_q(C^p)$ denotes the homology on the q-th dimension of the cochain complex C^p.

Note that if we *restrict* to the image of d for some $\sigma \in N_p(\mathcal{U})$, the map

$$\bigoplus_{i=0}^{p} H^q(X_{\partial_i \sigma}) \to H^q(X_\sigma)$$

has to be the direct sum of the induced morphisms in cohomology from the inclusions $X_\sigma \hookrightarrow X_{\partial_i \sigma}$.

Then we can obtain the $E_2^{p,q}$, and it can be shown that $E_r^{*,*}$ converges to $H^*(X)$. This is called the **Mayer-Vietoris spectral sequence** for X associated to the covering $\mathcal{U}$.

An example for this spectral sequence will be presented in Section 10.1; it is the computation of the group cohomology $H^*(\Gamma_1; \mathbb{Z})$, where Γ_1 is a Bianchi group.

3 Proper actions

We first review the definitions given in [4]; these will serve only as useful illustrations, since later in the thesis we will use the definitions given by Lück ([16], [17]), which are more oriented to the context of discrete groups.

Let G be a topological group. A space X with a continuous action $G \times X \to X$ is called a **G-space**. A **G-map** between two G-spaces X and Y is a continuous map $f : X \to Y$ such that $g \cdot f(x) = f(g \cdot x)$, for all $g \in G$, $x \in X$, which means that it is **G-equivariant**.

Definitions from Baum–Connes–Higson

Let G be a second countable, locally compact, Hausdorff topological group and X a G-space. Assume X and $G\backslash X$ are metrizable.

Definition 3.1. *For a G-space X, we say that the action is **proper** if for all $p \in X$ there exist an open neighbourhood U of p, a compact subgroup H of G, and a map ρ such that*

- *for all $g \in G$ and $u \in U$, $g \cdot u \in U$; and*
- *$\rho : U \to G/H$ is a G-map.*

For example, let H be a compact subgroup of G and let S be an H-space. Then $G \times_H S = (G \times S)/\sim$, where $(gh, s) \sim (g, h \cdot s)$, has a natural G-action. This space is sometimes known as the induction of S and it is a proper G-space.

Two G-maps $f_0,\ f_1 : X \to Y$ are called G**-homotopic** if they are homotopic through G-maps. This means that there exists a continuous map $f : X \times [0,1] \to Y$, where each map $f(\cdot, t) : X \to Y$ is a G-map for every $t \in [0,1]$, with $f(\cdot, 0) = f_0$ and $f(\cdot, 1) = f_1$.

Definition 3.2. *A **universal example for proper actions** of G, denoted $\underline{E}G$, is a proper G-space that satisfies the following universality property:*

- *If X is any proper G-space, there exists a G-map $f : X \to \underline{E}G$ which is unique up to G-homotopy.*

From the definition, it is clear that $\underline{E}G$ is unique up to G-homotopy.

There are a couple of explicit constructions for this space, depending on the structure of the group, but a universal example for proper actions always exists.

The following is a very useful characterization for this space.

Proposition 3.3. *A proper G-space Y is universal if and only if the following hold:*

- *If H is any compact subgroup of G, then there exists $y \in Y$ with $h\,y = y$ for all $h \in H$.*
- *Considering $Y \times Y$ as a G-space with the diagonal action $g(y_0, y_1) = (gy_0, gy_1)$, the two projections $\rho_0,\ \rho_1 : Y \times Y \to Y$ are G-homotopic.*

See [4] for proofs, examples, and more details.

Definitions from Lück

Let G be a locally compact, Hausdorff topological group. We define a **G-CW-complex** X as a G-space together with a G-invariant filtration

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset \bigcup_{n\geq 1} X_n = X,$$

where X has the colimit topology with respect to the filtration (which means that a subset $U \subset X$ is closed if and only if $U \cap X_n$ is closed in X_n for all n), and X_n is obtained from a pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times \mathbb{S}^{n-1} & \xrightarrow{\coprod_{i\in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times \mathbb{D}^n & \xrightarrow{\coprod_{i\in I_n} Q_i^n} & X_n \end{array}$$

attaching equivariant n-cells. The space X_n is called the **n-skeleton**.

Recall that a map $f : X \to Y$ is called proper if the preimage of every compact set in Y is compact in X; or, equivalently, if f is closed and $f^{-1}(y)$ is compact for any $y \in Y$.

From [16] and [17] we conclude the following.

Theorem 3.4. *The following three conditions are equivalent definitions for a* ***proper*** *G-CW-complex X:*

(i) *For each pair of points $x, y \in X$ there are open neighbourhoods V_x and V_y such that the closure of $\{g \in G : (gV_x) \cap V_y \neq \emptyset\}$ in G is compact.*

(ii) *The map $\theta : G \times X \to X \times X$, $(g, x) \mapsto (x, gx)$, is proper.*

(iii) *All the isotropy groups G_x, $x \in X$, are compact in G.*

Definition (iii) will be the most convenient for us to use.

Now, the **classifying space for proper actions** $\underline{E}G$ is defined the same as we previously did for a universal example for proper actions. Here, in the case of CW-complexes, there is a better homotopy characterization for $\underline{E}G$, as stated below.

Theorem 3.5. *A G-CW-complex X is a model for $\underline{E}G$ if and only if all its isotropy groups are compact in G and, for each $H \subset G$ compact, the H-fixed point set X^H is weakly contractible.*

4 Bredon cohomology

We define Bredon homology and cohomology as in [25].

Let G be a group with a family of subgroups $\mathfrak{F}$, closed under conjugation and finite intersections. Define the **orbit category** $\mathcal{O}_{\mathfrak{F}}G$ as the category whose objects are the sets G/H, for $H \in \mathfrak{F}$, and morphisms are G-maps $f_g : G/H \to G/K$, determined by an element $gK \in G/K$ such that $g^{-1}Hg \subset K$, so that it sends the coset H to the coset gK.

Denote $\mathcal{A}\mathbf{b}$ for the category of abelian groups, or $\mathbb{Z}$-modules. A **Bredon module** is defined to be a functor $M : \mathcal{O}_{\mathfrak{F}}G \to \mathcal{A}\mathbf{b}$, could be covariant or contravariant. A morphism $\Psi : M \to N$ between Bredon modules is a natural transformation; this means that for each $H \in \mathfrak{F}$ there is a morphism of abelian groups

$$\Psi(G/H) : M(G/H) \to N(G/H),$$

and these commute with the images by M and N of any morphism in $\mathcal{O}_{\mathfrak{F}}G$.

If M and N are both covariant, or contravariant, the group structure in each of the $\mathrm{Hom}(M(G/H), N(G/H))$ induces an abelian group structure in the set of natural transformations $\mathrm{mor}(M, N)$.

Also, if M is contravariant and N is covariant, we define an abelian group

$$M \otimes_{\mathfrak{F}} N = \bigoplus_{H \in \mathfrak{F}} M(G/H) \otimes_{\mathbb{Z}} N(G/H) \Big/ \sim$$

where for each $f : G/H \to G/K$, $m \in M(G/K)$, and $n \in N(G/H)$, we identify $M(f)(m) \otimes n$ with $m \otimes N(f)(n)$.

Let X be a G-CW-complex. Its cellular chain complex is denoted by $C_*(X)$. For each n, we can define a contravariant Bredon module given by

$$\underline{C_n(X)} : G/H \longmapsto C_n(X^H),$$

where X^H is the subspace fixed by the subgroup H. Let $\{\delta_\alpha\}$ be the n-cells of X, then we know that $C_n(X^H) \cong \bigoplus_\alpha \mathbb{Z}[\delta_\alpha^H]$, where δ_α^H means δ_α, if the cell is fixed by H, or empty, in which case it does not count in the sum. For a morphism $f_g : G/H \to G/K$ we have

$$\underline{f_g} := \underline{C_n(X)}(f_g) : C_n(X^K) \to C_n(X^H), \quad \delta_\alpha^K \longmapsto g \cdot \delta_\alpha^K =: \delta_{\alpha_g}^H.$$

There is a boundary map $\partial : C_n(X^H) \to C_{n-1}(X^H)$ for each $H \in \mathfrak{F}$, so this induces a boundary map

$$\partial : \underline{C_n(X)} \longrightarrow \underline{C_{n-1}(X)}.$$

This is well defined, since the commutativity of the diagram

$$\begin{tikzcd} C_n(X^K) \arrow[r, "\partial"] \arrow[d, "\underline{f_g}"] & C_{n-1}(X^K) \arrow[d, "\underline{f_g}"] \\ C_n(X^H) \arrow[r, "\partial"] & C_{n-1}(X^H) \end{tikzcd}$$

is obtained from the commutativity of the G-action in the cells of X and the boundary map in the cellular chain complex.

Let M and N be contravariant and covariant Bredon modules, respectively. We have chain complexes

$$\mathrm{mor}(\underline{C_*(X)}, M) \quad \text{and} \quad \underline{C_*(X)} \otimes_{\mathfrak{F}} N,$$

and we define the **Bredon cohomology** and **homology** groups with coefficients in M and N, respectively, as

$$\mathcal{H}_G^n(X; M) = H^n(\mathrm{mor}(\underline{C_*(X)}, M))$$

and

$$\mathcal{H}_n^G(X; N) = H_n(\underline{C_*(X)} \otimes_{\mathfrak{F}} N).$$

Now, let $K \in \mathfrak{F}$. We define the standard projective contravariant Bredon module P_K given as

$$P_K(G/H) = \mathbb{Z}[\mathrm{mor}(G/H, G/K)], \qquad \text{for } H \in \mathfrak{F},$$

and for a morphism $f : G/H_1 \to G/H_2$, the morphism $P_K(f) : P_K(G/H_2) \to P_K(G/H_1)$ is the linear extension of pre-composing with f.

For this module and any other contravariant module M, we have the isomorphism of abelian groups

$$ev_K : \mathrm{mor}(P_K, M) \longrightarrow M(G/K), \qquad \varphi \mapsto ev_K(\varphi) = \varphi(G/K)(1).$$

The inverse homomorphism is the one that sends an $x \in M(G/K)$ to the natural transformation given, in each G/H, as the linear extension of the map

$$\mathrm{mor}(G/H, G/K) \longrightarrow M(G/H), \qquad \lambda \mapsto M(\lambda)(x).$$

This isomorphism, $\mathrm{mor}(P_K, M) \cong M(G/K)$, may be interpreted as the Yoneda Lemma in category theory.

In a similar manner, if N is a covariant Bredon module, we will have an isomorphism

$$P_K \otimes_{\mathfrak{F}} N \cong N(G/K).$$

See [22] for more information on these isomorphisms.

Now, as before, let $\{\delta_\alpha\}$ be the n-cells of X, and let $\{e_\beta\}$ be a set of G-representatives of those n-cells; we know that

$$C_n(X^H) \cong \bigoplus_\alpha \mathbb{Z}[\delta_\alpha^H] \cong \bigoplus_\beta \mathbb{Z}[(G \cdot e_\beta)^H].$$

Besides, there is a ge_β fixed by H if and only if that g is such that $g^{-1}Hg \subset S_\beta$, where S_β is the stabilizer of the cell e_β, and the g's are taken as representatives in G/S_β, so we have a bijective correspondence

$$(G \cdot e_\beta)^H = \mathrm{mor}(G/H, G/S_\beta).$$

Therefore, we obtain

$$C_n(X^H) \cong \bigoplus_\beta \mathbb{Z}[\mathrm{mor}(G/H, G/S_\beta)] = \bigoplus_\beta P_{S_\beta}(G/H),$$

so, as Bredon modules, $\underline{C_n(X)} \cong \bigoplus_\beta P_{S_\beta}$.

With Bredon modules, in the same way as with $\mathbb{Z}$-modules, the morphisms from a direct sum to another module is the direct product of the sets of morphisms from each in the direct sum to the other module. Then, we have an isomorphism of chain complexes

$$\mathrm{mor}_G(\underline{C_*(X)}, M) \cong \prod_{\beta^*} \mathrm{mor}(P_{S_{\beta^*}}, M) \cong \prod_{\beta^*} M(G/S_{\beta^*}),$$

where $\{\beta^*\}$ indexes the G-representatives of $*$-cells. This becomes a direct sum assuming there are finite representatives for the cells.

4.1 Coefficients in the representation ring

We will use the contravariant Bredon module $\mathcal{R}$ sending G/H to $R(H)$, the representation ring of the subgroup H. The morphisms are obtained from the composition of restriction and the isomorphism given by conjugation: for any $f_g : G/H \to G/L$ the morphism $\mathcal{R}(f_g)$ is the composition

$$R(L) \xrightarrow{\mathrm{Res}^L_{g^{-1}Hg}} R(g^{-1}Hg) \xrightarrow{\cong} R(H).$$

Then, as seen before, we have the isomorphism

$$\mathrm{mor}_G(\underline{C_n(X)}, \mathcal{R}) \cong \bigoplus_{\alpha} R(S_\alpha),$$

given that there are finite orbit representatives of n-cells. Here, the differential is given by restriction of representations, from the stabilizer of an n-cell to the stabilizer of the corresponding $(n+1)$-cell that contains it.

Similarly, we can consider $\mathcal{R}$ as a covariant Bredon module, setting $\mathcal{R}(f_g)$ to be the composition

$$R(H) \xrightarrow{\cong} R(g^{-1}Hg) \xrightarrow{\mathrm{Ind}^L_{g^{-1}Hg}} R(L).$$

Then we have the chain complex

$$\underline{C_n(X)} \otimes_{\mathfrak{F}} \mathcal{R} \cong \bigoplus_{\alpha} R(S_\alpha),$$

where the differential is given by induction of representations.

The following is an important result from [22] regarding conjugacy classes of a group and the 0-th Bredon homology group associated. We will use it as a check for the computations to the Hecke operator.

Theorem 4.1. *Let G be any group and let* $\mathrm{FC}(G)$ *be the set of conjugacy classes of elements of finite order in G. There is an isomorphism*

$$\mathcal{H}_0^G(\underline{E}G;\mathcal{R}) \otimes_{\mathbb{Z}} \mathbb{C} \;\cong\; \mathbb{C}[\mathrm{FC}(G)].$$

In particular, if $\mathcal{H}_0^G(\underline{E}G;\mathcal{R})$ is a free abelian group, its rank is equal to the number of conjugacy classes of elements of finite order in G.

5 K-theory

First we recall the basic, classical definitions needed for K-theory, which are given for compact, Hausdorff spaces. Then, we give the definitions developed by Lück and Oliver for finite proper G-CW-complexes, with G discrete, the ones we are interested in. Everything is done over the complex numbers.

Let X be a topological space (compact, Hausdorff). A **vector bundle over** X is a space E together with a map $p : E \to X$ such that for any $x \in X$

- $p^{-1}(x) = E_x$ has a vector space structure over $\mathbb{C}$, compatible with the topology given from E, and
- there is a neighbourhood $U \subset X$ of x for which the preimage $p^{-1}(U)$ is isomorphic to the space $\mathbb{C}^n \times U$, for some integer n, where $p^{-1}(y) = E_y$, for any $y \in U$, is identified with $\mathbb{C}^n \times \{y\}$ as an isomorphism of vector spaces.

The second condition is known as local triviality.

Here, X is called the **base space**, E the **total space**, p the **projection map**, and, for each $x \in X$, E_x is called the **fibre over** x. We may refer to a vector bundle using both E and p or just with the total space E. If the number n is constant over all X we say that E is a vector bundle of **dimension** n.

Given two vector bundles $p : E \to X$ and $q : F \to X$. A continuous map $\varphi : E \to F$ is said to be a **homomorphism of vector bundles** if

- $q \circ \varphi = p$, and
- for any $x \in X$, $\varphi : E_x \to F_x$ is a linear transformation.

The map φ is an **isomorphism** if it is a homeomorphism, in which case we write simply $E \cong F$. A vector bundle is called **trivial bundle** if it is globally trivial, i.e., it is isomorphic to the vector bundle $\mathbb{C}^n \times X$, for some n.

If there is a continuous map $f : Y \to X$ and a vector bundle $p : E \to X$, there is a **pullback** vector bundle $f^*(p) : f^*(E) \to Y$, where

$$f^*(E) = \{(e, y) \in E \times Y \ : \ p(e) = f(y)\}$$

and $f^*(p)$ is the projection to Y. Note that if f is an inclusion, the pullback is the same as the **restriction** of p to the subspace Y; this is usually denoted $E|_Y$ or $E|Y$.

Having two vector bundles E and F over a space X, we can define the direct sum $E \oplus F$ and the tensor product $E \otimes F$, defining each fibre to be the direct sum and tensor product of the fibres, respectively. This operations are commutative, and the tensor product distributes over direct sum, modulo isomorphism. For detailed demonstrations see [2].

There is a well known construction of an abelian group G up from an abelian semigroup A, named after Grothendieck. The simplest way to describe this group G is as the set of formal differences $a - b$ making the identification $a_1 - b_1 \sim a_2 - b_2$ if there exists $c \in A$ such that $a_1 + b_2 + c = a_2 + b_1 + c$.

Now, if we define $(\mathrm{Vect}(X), \oplus)$ to be the set of isomorphism classes of vector bundles over X, then it is an abelian semigroup. Note that any map $X \to Y$ induces a map $\mathrm{Vect}(Y) \to \mathrm{Vect}(X)$. Also, an elementary fact here is that if the map is a homotopy equivalence, then the map $\mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ is a bijection.

Applying the construction of the Grothendieck group, we obtain and define the **K-theory group of** X, $K(X)$. In this case, $\mathrm{Vect}(X)$ is also a semiring, and $K(X)$ will have a commutative ring structure as well, but we will not use this.

Example. We give an outline of the computation for the K-theory group of the sphere $\mathbb{S}^2$.

First, we consider the sphere as the union of two hemispheres, whose intersection is a copy of $\mathbb{S}^1$. Restricted to each hemisphere, any vector bundle is trivial, since these are contractible, so we can determine a vector bundle over $\mathbb{S}^2$ by how the (trivial) bundles, necessarily of same dimension, on each hemisphere are joined along the circle. It can be proved that any vector bundle, modulo isomorphism, is uniquely determined by a continuous function $f : \mathbb{S}^1 \to \mathrm{GL}_n(\mathbb{C})$, modulo homotopy. The associated bundle can be defined as

$$E_f \;=\; \mathbb{D}^2_- \times \mathbb{C}^n \sqcup \mathbb{D}^2_+ \times \mathbb{C}^n \;/ \sim,$$

where $(x, v) \in \partial\,\mathbb{D}^2_- \times \mathbb{C}^n$ is identified with $(x, f(x)(v)) \in \partial\,\mathbb{D}^2_+ \times \mathbb{C}^n$. Then, for each n, there is a bijection from the set of homotopy classes of maps $f : \mathbb{S}^1 \to \mathrm{GL}_n(\mathbb{C})$ and the set of isomorphism classes of vector bundles over $\mathbb{S}^2$ of dimension n.

Furthermore, with the one-dimensional bundles 1 (trivial) and the canonical line bundle H, given *canonically* from the identification $\mathbb{S}^2 = \mathbb{C}\,\mathrm{P}^1$, it can be proved that $(H \otimes H) \oplus 1 \cong H \oplus H$, or, written as in $K(\mathbb{S}^2)$, $(H - 1)^2 = 0$. Later, we get an isomorphism of rings

$$\mathbb{Z}[H]/(H - 1)^2 \longrightarrow K(\mathbb{S}^2).$$

In particular, as a group, we have $K(\mathbb{S}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

There is a way to obtain a cohomology theory $K^*(-)$ in the category of compact, Hausdorff spaces; see [2] for the complete definitions that give rise to this.

5.1 Equivariant K-theory

Let G be a topological group and X be a G-space. A G-space E is a **G-vector bundle over X** if

- E is a vector bundle over X,
- the projection map $E \to X$ is a G-map, and
- for each $x \in X$ and $g \in G$, the map $E_x \to E_{gx}$, given by the action of g, is a linear transformation.

A G-CW-complex is called **finite** if it has finitely many orbits of cells. A **G-CW-pair** is a pair of G-spaces (X, A), where X is a G-CW-complex and A is a G-invariant subcomplex.

Definition 5.1. *For any discrete group G and any finite proper G-CW-complex X, let $K_G(X) = K^0_G(X)$ be the Grothendieck group of the semigroup $\mathrm{Vect}_G(X)$ of isomorphism classes of G-vector bundles over X. Define $K^{-n}_G(X)$, for all $n > 0$, by setting*

$$K^{-n}_G(X) = \mathrm{Ker}(\, K_G(X \times \mathbb{S}^n) \xrightarrow{\mathrm{incl}^*} K_G(X)\,).$$

For any proper G-CW-pair (X,A), and $n \geq 0$, set

$$K_G^{-n}(X,A) = \mathrm{Ker}(\, K_G^{-n}(X \cup_A X) \xrightarrow{i_2^*} K_G^{-n}(X)\,).$$

And, let $K_G^n(X) = K_G^{-n}(X)$ and $K_G^n(X,A) = K_G^{-n}(X,A)$.

With these definitions, $K_G^*(-)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded multiplicative equivariant cohomology theory in the category of finite proper G-CW-complexes. See [19] for further information.

Now, we give the isomorphism that will allow us to define a Hecke operator in K-theory.

First, note that there is a natural action of a group G over the K-theory of a G-space X, given by the pullback of the action on the space. Furthermore, for any $g \in G$, there is an action of the centralizer $C(g) \subset G$ on the fixed (point set) space X^g, and thus on its K-theory. In this way, by $K^*(X^g)^{C(g)}$ we mean the subgroup of $K^*(X^g)$ fixed by the action of $C(g)$.

Theorem 5.2. *Let G be a discrete group and X a finite proper G-CW-complex, then*

$$K_G^*(X) \otimes \mathbb{C} \cong \bigoplus_{[g]} K^*(X^g)^{C(g)} \otimes \mathbb{C}$$

where $[g]$ runs over conjugacy classes of elements of finite order in G.

This isomorphism is described by Atiyah and Segal in [3] for a finite group and a compact manifold and is given explicitly on K-theory groups in [7] by, with $\pi : E \to X$ vector bundle,

$$[E] \longmapsto \bigoplus_{[g]} \bigoplus_{\lambda \in \mathbb{S}^1} [\pi(E|_{X^g})_\lambda] \otimes \lambda,$$

where $\pi(E|_{X^g})_\lambda$ denotes the vector bundle of λ-eigenvectors considering the action of the element g over $\pi(E|_{X^g})$.

But, reviewing through the proof in [3] and considering the wide results obtained by Lück and Oliver in [19] and [18] for discrete groups and finite proper G-CW-complexes, where the definitions of K-theory are given in terms of vector bundles as well, we can state the version in Theorem 5.2.

5.2 The Atiyah–Hirzebruch spectral sequence

The well known Atiyah–Hirzebruch spectral sequence lets us compute a (co)homology theory in terms of the ordinary (co)homology using the (co)homology theory applied to a point as coefficients.

Mislin [22] and Sánchez-García [25] describe this spectral sequence for Bredon homology and equivariant K-homology. Here we will use the version in equivariant K-theory mentioned by Lück and Oliver in [18].

For a discrete group G and any finite dimensional proper G-complex X, the skeletal filtration of $K_G^*(X)$ induces a spectral sequence

$$E_2^{p,2q} \cong \mathcal{H}_G^p(X; R(-)) \Longrightarrow K_G^*(X).$$

Then, if $\dim(X) = 2$, which will be our case, we have that Bredon cohomology is trivial for $p > 2$, so the spectral sequence collapses in E_2. With this, there is a short exact sequence

$$0 \longrightarrow \mathcal{H}_G^2(X; \mathcal{R}) \longrightarrow K_G^0(X) \longrightarrow \mathcal{H}_G^0(X; \mathcal{R}) \longrightarrow 0,$$

and $K_G^1(X) = \mathcal{H}_G^1(X; \mathcal{R})$.

6 Hecke operators

Here we will give an idea of how Hecke operators are described in general, the basic definitions and the interpretations in other contexts. For the first part we follow Shimura's treatment of Hecke operators [28]. Classic aspects of the theory in the context of modular forms are covered by Diamond and Shurman [8]. For the action of Hecke operators on the K-theory of Bianchi groups, we build upon the results of Mesland and Şengün [21].

Let G be a group. Two subgroups of G are said to be **commensurable** if their intersection has finite index in both. Commensurability defines an equivalence relation in the set of subgroups of G. If Γ_1 and Γ_2 are subgroups of G which are commensurable, we use the notation

$$\Gamma_1 \sim \Gamma_2.$$

We define the **commensurator of Γ in G** as the subgroup

$$\widetilde{\Gamma} = \{\ g \in G\ :\ \Gamma \sim g\Gamma g^{-1}\}.$$

Note that if Γ_1 and Γ_2 are commensurable subgroups of G, then

$$\widetilde{\Gamma_1} = \widetilde{\Gamma_2}.$$

In examples of arithmetic relevance, the group G will be a Lie group defined as the group of real or complex points of an algebraic group defined over $\mathbb{Q}$, and Γ will be a discrete subgroup. For instance, we have $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{PGL}_2^+(\mathbb{R})$ acting on the hyperbolic plane $\mathbb{H}^2$, and $\mathrm{PSL}_2(\mathbb{Z}[i]) \subset \mathrm{PGL}_2(\mathbb{C})$ acting on the hyperbolic space $\mathbb{H}^3$.

Example 6.1. *Let Γ be the modular group $\mathrm{PSL}_2(\mathbb{Z})$ viewed as a subgroup of $\mathrm{PGL}_2^+(\mathbb{R})$, then the commensurator of Γ in $\mathrm{PGL}_2^+(\mathbb{R})$ is*

$$\widetilde{\Gamma} = \mathrm{PGL}_2^+(\mathbb{Q}).$$

Example 6.2. *Let $K \subset \mathbb{C}$ be a quadratic imaginary extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$. Let Γ be the corresponding Bianchi group $\mathrm{PSL}_2(\mathcal{O}_K)$ viewed as a subgroup of $\mathrm{PGL}_2(\mathbb{C})$, then the commensurator of Γ in $\mathrm{PGL}_2(\mathbb{C})$ is*

$$\widetilde{\Gamma} = \mathrm{PGL}_2(K).$$

6.1 Double cosets

Let G be a group and let Γ_1 and Γ_2 be two commensurable subgroups of G. Given an element g in their commensurator $\widetilde{\Gamma_1} = \widetilde{\Gamma_2}$ we consider the doble coset in G given by

$$\Gamma_1\, g\, \Gamma_2.$$

The left action of Γ_1 on the double coset $\Gamma_1 g \Gamma_2$ has a finite number of orbits. To compute this number, let $\Gamma_{2,1} = \Gamma_2 \cap g^{-1}\Gamma_1 g$ and notice that the map

$$\begin{array}{ccc} \Gamma_2 & \longrightarrow & \Gamma_1 g \Gamma_2 \\ \gamma_2 & \longmapsto & g\gamma_2 \end{array}$$

induces a surjection from Γ_2 to the quotient $\Gamma_1\backslash\Gamma_1 g\Gamma_2$ and also gives a bijection from $\Gamma_{2,1}\backslash\Gamma_2$ to $\Gamma_1\backslash\Gamma_1 g\Gamma_2$, then there is a decomposition

$$\Gamma_1 g \Gamma_2 = \bigsqcup_{i=1}^{d} \Gamma_1 \alpha_i, \qquad \text{where} \qquad d = [\Gamma_2 : \Gamma_{2,1}].$$

Analogously, taking $\Gamma_{1,2} = \Gamma_1 \cap g\Gamma_2 g^{-1}$ we obtain a decomposition

$$\Gamma_1 g \Gamma_2 = \bigsqcup_{j=1}^{e} \beta_j \Gamma_2, \qquad \text{where} \qquad e = [\Gamma_1 : \Gamma_{1,2}].$$

These decompositions lead to a natural product of double cosets. If Γ_1, Γ_2, and Γ_3 are commensurable subgroups of G and g, h are elements in their commensurator $\widetilde{\Gamma_1}$ with

$$\Gamma_1 g \Gamma_2 = \bigsqcup_{i} \Gamma_1 \alpha_i \qquad \text{and} \qquad \Gamma_2 h \Gamma_3 = \bigsqcup_{j} \beta_j \Gamma_3,$$

we have

$$(\Gamma_1 g \Gamma_2) \cdot (\Gamma_2 h \Gamma_3) = \bigcup_{i,j} \Gamma_1 \alpha_i \beta_j \Gamma_3.$$

So, after omitting repetitions, this is a disjoint union of double cosets $\Gamma_1 \xi \Gamma_3$. To do this keeping record of those repetitions, we think the cosets inside the free abelian group generated by the double cosets.

Let us suppose the subgroups Γ_k are contained in a semigroup Δ that is contained in their commensurator in G. We denote by

$$\mathfrak{R}(\Gamma_k, \Gamma_l; \Delta)$$

the free abelian group generated by double cosets of the form $\Gamma_k g \Gamma_l$ with $g \in \Delta$. Now we define the product

$$(\Gamma_1 g \Gamma_2) \cdot (\Gamma_2 h \Gamma_3) = \sum_{\delta} c_{g,h}^{\delta} \Gamma_1 \delta \Gamma_3,$$

where $c_{g,h}^{\delta}$ is the number of pairs of indices (i, j) such that $\Gamma_1 \alpha_i \beta_j = \Gamma_1 \delta$. The linear extension of this product becomes a bilinear map of $\mathbb{Z}$-modules

$$\mathfrak{R}(\Gamma_1, \Gamma_2; \Delta) \times \mathfrak{R}(\Gamma_2, \Gamma_3; \Delta) \longrightarrow \mathfrak{R}(\Gamma_1, \Gamma_3; \Delta)$$

which is associative in the obvious sense. In particular, given a subgroup Γ of G and a subsemigroup of G with $\Gamma \subseteq \Delta \subseteq \widetilde{\Gamma}$, the group

$$\mathfrak{R}(\Gamma; \Delta) = \mathfrak{R}(\Gamma, \Gamma; \Delta)$$

becomes a ring, which will be called the **Hecke ring of Γ with respect to Δ.**

6.2 Action on group cohomology

Let Γ_1 and Γ_2 be two commensurable subgroups of a group G and let Δ be a subsemigroup of their commensurator $\widetilde{\Gamma}$ with $\Gamma_i \subseteq \Delta$ for $i = 1, 2$. Let M be an abelian group on which Δ acts. We can consider M as a left Γ_i-module, $i = 1, 2$.

The elements of $\mathfrak{R}(\Gamma_1, \Gamma_2; \Delta)$ define homomorphisms from the cohomology groups of Γ_1 with coefficients in M to the cohomology groups of Γ_1 with coefficients in M. These operators are called **Hecke operators associated to** $(\Gamma_1, \Gamma_2; \Delta)$.

First, write

$$\Gamma_1 g \Gamma_2 = \bigsqcup_{i=1}^{d} \Gamma_1 \alpha_i, \qquad \alpha_i \in \Delta,$$

and let $m \in M^{\Gamma_1}$ be an element of M fixed by Γ_1. We define the element

$$m \,|\, \Gamma_1 g \Gamma_2 = \sum_{i=1}^{d} \alpha_i^{-1} \cdot m$$

in M; it is independent from the representatives α_i and it is fixed by Γ_2, so the coset $\Gamma_1 g \Gamma_2$ defines a map

$$T_g \;:\; M^{\Gamma_1} \longrightarrow M^{\Gamma_2}.$$

These maps can be extended linearly to define operators associated to all elements of $\mathfrak{R}(\Gamma_1, \Gamma_2; \Delta)$. Indeed, if

$$\xi = \sum_{k=1}^{r} c_k (\Gamma_1 g_k \Gamma_2) \;\in\; \mathfrak{R}(\Gamma_1, \Gamma_2; \Delta)$$

and $m \in M^{\Gamma_1}$, we have $T_\xi : M^{\Gamma_1} \to M^{\Gamma_2}$ given by

$$T_\xi(m) = \sum_{k=1}^{r} c_k \, T_{g_k}(m) = \sum_{k=1}^{r} c_k (m | \Gamma_1 g_k \Gamma_2).$$

In the case of $\Gamma_1 = \Gamma_2 = \Gamma$, we have an action of the Hecke ring $\mathfrak{R}(\Gamma; \Delta)$ on M^Γ.

The n-th cohomology group of Γ with coefficients on the left Γ-module M is defined as the image of M under the n-th right derived functor of

$$M \longmapsto M^\Gamma \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

so the operators T_ξ extend to Hecke operators between cohomology groups:

$$T_\xi : H^n(\Gamma_1; M) \longrightarrow H^n(\Gamma_2; M).$$

An alternative way to define the action of the Hecke ring on group cohomology is to use that the cohomology group $H^n(\Gamma; M)$ can be computed using the standard complex $\mathcal{C}^* = \mathcal{C}^*(\Gamma; M)$. An (homogeneous) n-cochain $\phi \in \mathcal{C}^n$, $n \geq 0$, is defined as a function

$$\phi \,:\, \underbrace{\Gamma \times \cdots \times \Gamma}_{n+1} \longrightarrow M$$

satisfying

$$\phi(\alpha\gamma_0, \alpha\gamma_1, \ldots, \alpha\gamma_n) \;=\; \alpha\, \phi(\gamma_0, \gamma_1, \ldots, \gamma_n)$$

for all $\alpha, \gamma_0, \gamma_1, \ldots, \gamma_n \in \Gamma$.

The coboundary map $\mathrm{d} : \mathcal{C}^n \to \mathcal{C}^{n+1}$ of the complex is given by

$$\mathrm{d}_n\phi\,(\gamma_0, \gamma_1, \ldots, \gamma_{n+1}) \;=\; \sum_{j=0}^{n+1} (-1)^j\, \phi(\gamma_0, \ldots, \widehat{\gamma}_j, \ldots \gamma_{n+1})$$

where the notation $\widehat{\gamma}_j$ indicates that the j-th term has been omitted. A straightforward computation gives $\mathrm{d}_{n+1} \circ \mathrm{d}_n = 0$ for all $n \geq 0$.

The cohomology of the complex $\mathcal{C}^*$ computes the groups $H^n(\Gamma; M)$, that is,

$$H^n(\Gamma; M) = \mathrm{Ker}(\mathrm{d}_n)/\mathrm{Im}(\mathrm{d}_{n-1}).$$

As above, with Γ_1 and Γ_2 two commensurable subgroups of a group G and Δ a subsemigroup of $\widetilde{\Gamma}$ with $\Gamma_i \subset \Delta$, $i = 1, 2$, if $g \in \Delta$ we have

$$\Gamma_1 g \Gamma_2 = \bigsqcup_{j=1}^{d} \Gamma_1 \alpha_j, \qquad \alpha_j \in \Delta.$$

For $\gamma \in \Gamma_2$, we denote by σ_g^γ the unique permutation of $\{1, \ldots, d\}$ for which

$$\Gamma_1 \alpha_j \gamma = \Gamma_1 \alpha_{\sigma_\gamma^g(j)},$$

then for each $j = 1, \ldots, d$ we obtain a map

$$\rho_j^g \,:\, \Gamma_2 \longrightarrow \Gamma_1$$

where for $\gamma \in \Gamma_2$ the element $\rho_j^g(\gamma) \in \Gamma_1$ is determined by $\alpha_j \gamma = \rho_j^g(\gamma) \alpha_{\sigma_\gamma^g(j)}$.

Now, let M be an abelian group on which Δ acts by endomorphisms and let g be an element of Δ. We define the image of an n-cochain $\phi \in \mathcal{C}^n(\Gamma_1; M)$ under the action of a double coset $\Gamma_1 g \Gamma_2$ as an element of $\mathcal{C}^n(\Gamma_2; M)$ given by

$$(\phi \,|\, \Gamma_1 g \Gamma_2)(\gamma_0, \ldots, \gamma_n) = \sum_{j=1}^{d} \alpha_j^{-1}\, \phi(\rho_j^g(\gamma_0), \ldots, \rho_j^g(\gamma_n)).$$

Also, this slash operator $\cdot \,|\, \Gamma_1 g \Gamma_2$ commutes with the differentials, so it induces a well defined morphism in cohomology

$$T_g \,:\, H^n(\Gamma_1; M) \longrightarrow H^n(\Gamma_2; M)$$

As before, we can extend linearly and obtain a morphism

$$T_\xi \,:\, H^n(\Gamma_1; M) \longrightarrow H^n(\Gamma_2; M)$$

for any $\xi \in \mathfrak{R}(\Gamma_1, \Gamma_2; \Delta)$.

In particular, in the case $\Gamma = \Gamma_1 = \Gamma_2$ we have an action of the Hecke ring $\mathfrak{R}(\Gamma; \Delta)$ on $H^n(\Gamma; M)$, hence it is a $\mathfrak{R}(\Gamma; \Delta)$-module.

Further information on this action can be found in [15] and, together with its functorial properties and its relation to the classical theory of Hecke operators, in [14].

6.3 Hecke correspondences

Assume now that the group G acts on a topological space X and consider the action of the subgroups Γ_i on X. We will be interested in the case where G is a Lie group, the groups Γ_1 and Γ_2 are commensurable discrete subgroups of G and X is a homogeneous G-space. Also, we assume that the action of the discrete groups Γ_i on X satisfies sufficient conditions for the quotients X/Γ_i to be well behaved.

Given an element $g \in \widetilde{\Gamma_1} = \widetilde{\Gamma_2}$ consider the groups

$$\Gamma_{1,2} = \Gamma_1 \cap g \Gamma_2 g^{-1} \quad \text{and} \quad \Gamma_{2,1} = \Gamma_2 \cap g^{-1} \Gamma_1 g.$$

We have group morphisms

$$\begin{array}{ccc} \Gamma_{1,2} & \longrightarrow & \Gamma_{2,1} \\ \downarrow & & \downarrow \\ \Gamma_1 & & \Gamma_2 \end{array}$$

where the horizontal arrow is a group isomorphism (given by conjugation) and the vertical ones are inclusions with finite index. These morphisms induce maps between the corresponding quotients of X:

$$\begin{tikzcd} X/\Gamma_{1,2} \arrow[r] \arrow[d, two heads] & X/\Gamma_{2,1} \arrow[d, two heads] \\ X/\Gamma_1 & X/\Gamma_2 \end{tikzcd}$$

where the horizontal arrow is a homeomorphism and the vertical ones are finite index covers. This diagram determines a correspondence

$$\mathcal{C}_g \subset X/\Gamma_1 \times X/\Gamma_2$$

homeomorphic to $X/\Gamma_{1,2}$. This correspondence is called the **Hecke correspondence** from X/Γ_2 to X/Γ_1 associated to g.

Next, we outline an example of this correspondence given in divisors; this approach leads to the classical theory for congruence subgroups of the modular group, which is described in the next section.

The group $G = \mathrm{GL}_2^+(\mathbb{R})$ acts on the upper half plane $\mathbb{H}$ via Möbius transformations: for $z \in \mathbb{H}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Given a subgroup $\Gamma \subset \mathrm{GL}_2^+(\mathbb{R})$ commensurable with $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$, we consider the quotient

$$Y_\Gamma = \mathbb{H} \,/\, \Gamma$$

and its compactification

$$X_\Gamma = (\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}) \,/\, \Gamma.$$

These quotients admit natural complex structures, and the Riemman surfaces Y_Γ and X_Γ can be considered as algebraic curves over $\mathbb{C}$, these are the modular curves associated to Γ.

If Γ_1 and Γ_2 are two subgroups of $\mathrm{GL}_2^+(\mathbb{R})$ commensurable with $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ and $g \in \widetilde{\Gamma(1)} = \mathrm{GL}_2^+(\mathbb{Q})$ is an element in their commensurator, we have as before a correspondence between modular curves given by

$$\begin{tikzcd} X_{\Gamma_{1,2}} \arrow[r, "\varphi"] \arrow[d, two heads, "\pi_1"] & X_{\Gamma_{2,1}} \arrow[d, two heads, "\pi_2"] \\ X_{\Gamma_1} & X_{\Gamma_2} \end{tikzcd}$$

Now, for a point $p \in X_{\Gamma_2}$ let $\pi_2^{-1}(\{p\}) = \{t_1, \dots t_d\}$, but note that each t_i has a multiplicity e_i, according to its ramification degree, so we define

$$T_g(p) = \sum_{i=1}^{n} e_i \ \pi_1(\varphi^{-1}(t_i)),$$

which is an element of the free abelian group generated by the points of X_{Γ_1}, the divisor group $\mathrm{Div}(X_{\Gamma_1})$. Extending linearly we obtain a map

$$T_g \, : \, \mathrm{Div}(X_{\Gamma_2}) \longrightarrow \mathrm{Div}(X_{\Gamma_1})$$

that defines the Hecke operator T_g at the level of divisors.

6.4 The classical theory

The Hecke operators described in the previous section arose originally in the context of automorphic forms and automorphic functions for congruence subgroups of the modular group $\Gamma(1)$.

A subgroup $\Gamma \subset \mathrm{GL}_2^+(\mathbb{R})$ is a congruence subgroup of level $N \in \mathbb{N}$ if it contains the group

$$\Gamma(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \quad : \quad \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

with finite index. If Γ is a congruence subgroup, then it is necessarily a discrete subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ commensurable with $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.

Since the half plane $\mathbb{H}$ is simply connected any line bundle on $\mathbb{H}$ is trivial, and holomorphic line bundles on the modular curve

$$X_\Gamma = (\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}) \ / \ \Gamma$$

are obtained as quotients of the trivial bundle $\mathbb{H} \times \mathbb{C}$. Such holomorphic line bundles on the modular curve are classified by automorphy factors

$$j \, : \, \Gamma \times \mathbb{H} \longrightarrow \mathbb{C}^*,$$

which are holomorphic functions, for fixed $\gamma \in \Gamma$, and satisfy the cocycle condition $j(\gamma\delta, z) = j(\gamma, \delta z)j(\delta, z)$.

Given a congruence subgroup Γ and an integer k, a **Γ-automorphic form of weight** k is a meromorphic section of the line bundle on X_Γ corresponding to the automorphy factor

$$j(z, \gamma) = \frac{\det(\gamma)^{\frac{k}{2}}}{(cz+d)^k} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Equivalently, a Γ-automorphic form of weight k can be defined as a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f(z) = \frac{\det(\gamma)^{\frac{k}{2}}}{(cz+d)^k} f(\gamma z) \quad \text{for all} \quad \gamma \in \Gamma,$$

and is meromorphic at the cusps, i.e. the finite set of points $(\mathbb{Q} \cup \{i\infty\})/\Gamma \subset X_\Gamma$. A **$\Gamma$-automorphic function** is simply a Γ-automorphic form of weight cero.

We denote by $\mathcal{M}_k(\Gamma)$ the space of Γ-automorphic forms of weight k that are holomorphic on $\mathbb{H}$. The space $\mathcal{M}_k(\Gamma)$ is a finite dimensional vector space over $\mathbb{C}$, and the graded algebra $\mathcal{M}(\Gamma) = \bigoplus_k \mathcal{M}_k(\Gamma)$ plays a central role in number theory. In the case of $\Gamma \subset \Gamma(1)$, the elements of $\mathcal{M}(\Gamma)$ are called **modular forms of level** Γ.

Given an element $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\Gamma} = \mathrm{GL}_2^+(\mathbb{Q})$ and f a Γ-automorphic form of weight k, we use the notation

$$(f|_k\, \alpha)(z) = \frac{\det(\alpha)^{\frac{k}{2}}}{(cz+d)^k} f(\alpha z).$$

Now, let Γ_1 and Γ_2 be two congruence subgroups in $\mathrm{GL}_2^+(\mathbb{R})$ and let Δ be a sub-semigroup of $\mathrm{GL}_2^+(\mathbb{Q})$ with $\Gamma_i \subset \Delta$, $i = 1, 2$. For any $g \in \Delta$ we have a decomposition of the corresponding double coset

$$\Gamma_1 g \Gamma_2 = \bigsqcup_{i=1}^{d} \Gamma_1 \alpha_i, \qquad \alpha_i \in \Delta,$$

then, given an element $f \in \mathcal{M}_k(\Gamma_1)$, the function

$$f|_k\, \Gamma_1 g \Gamma_2 = \det(g)^{\frac{k}{2}-1} \sum_{i=1}^{d} f|_k\, \alpha_i$$

is a Γ_2-automorphic form of weight k holomorphic on $\mathbb{H}$, so we obtain in this way a Hecke operator

$$T_g : \mathcal{M}_k(\Gamma_1) \longrightarrow \mathcal{M}_k(\Gamma_2).$$

Viewing automorphic forms as sections of line bundles over modular curves, and having the correspondence showed earlier

$$\begin{tikzcd} X_{\Gamma_{1,2}} \arrow[r] \arrow[d, two heads] & X_{\Gamma_{2,1}} \arrow[d, two heads] \\ X_{\Gamma_1} & X_{\Gamma_2} \end{tikzcd}$$

we will have that the Hecke operator T_g is given by the composition of the pullback and pushforward on sections via these maps.

The relation between the classical theory and the results in group cohomology can be found in [14]. For an account of the corresponding theory of forms and Hecke operators for Bianchi groups see [27] and references therein.

6.5 In K-theory

Following the previous discussions and results, we want to give a definition for Hecke operators in K-theory in the form of pullback-pushforward, or in this case restriction-corestriction, as described in [21]. In this definition, the subgroup will just need to have finite index, but in our computations it will be an intersection $\Gamma \cap g\Gamma g^{-1}$ as seen before.

Let $H \subset G$, with $(G : H) < \infty$, and let X be a proper G-CW-complex. We will define the two maps

$$\mathrm{res} : K^*_G(X) \longrightarrow K^*_H(X) \quad \text{and} \quad \mathrm{cores} : K^*_H(X) \longrightarrow K^*_G(X)$$

through the isomorphism

$$K^*_G(X) \otimes \mathbb{C} \cong \bigoplus_{[g]} K^*(X^g)^{C(g)} \otimes \mathbb{C},$$

seen in Section 5.1, so we need to know how the conjugacy classes of finite elements in H relate with those in G.

Take a conjugacy class $[g]$ in G such that the set $[g] \cap H$ is not empty, then there are some $h_1, \ldots, h_n \in H$ for which

$$[g] \cap H = [h_1] \sqcup \cdots \sqcup [h_n].$$

For each one, let $h_i = \gamma_i^{-1} g \gamma_i$ with $\gamma_i \in G$. We know that $C_G(h_i) = \gamma_i^{-1}\, C_G(g)\, \gamma_i$, since if $a \in C_G(g)$ then $ag = ga$ and

$$(\gamma_i^{-1} a \gamma_i) h_i = \gamma_i^{-1} a g \gamma_i = \gamma_i^{-1} g a \gamma_i = h_i (\gamma_i^{-1} a \gamma_i).$$

Also, $C_H(h_i) = C_G(h_i) \cap H = (\gamma_i^{-1}\, C_G(g)\, \gamma_i) \cap H$.

Furthermore, there is a homeomorphism $X^{h_i} \to X^g$ given by the action of γ_i. Thus, for each i, there is a natural homomorphism

$$K^*(X^g)^{C_G(g)} \longrightarrow K^*(X^{h_i})^{C_H(h_i)},$$

that is well defined in invariants by the previous facts. Tensoring with $\mathbb{C}$ and adding up, we obtain a map

$$K^*(X^g)^{C_G(g)} \otimes \mathbb{C} \longrightarrow \bigoplus_{i=1}^{n} K^*(X^{h_i})^{C_H(h_i)} \otimes \mathbb{C}.$$

And since a conjugacy class in H corresponds to only one conjugacy class in G, we obtain the map

$$\mathrm{res} : \bigoplus_{[g] \text{ in } G} K^*(X^g)^{C_G(g)} \otimes \mathbb{C} \longrightarrow \bigoplus_{[h] \text{ in } H} K^*(X^h)^{C_H(h)} \otimes \mathbb{C}.$$

The restriction map can be thought just as restriction of the action on the vector bundles, but we use this definition to make the computations. Likewise, we give a definition for the corestriction map in the decomposition by conjugacy classes of finite elements that is useful for our computations.

To construct the corestriction map, we follow the definition of the induction on class functions, summing over conjugates. With the conjugacy classes as before, let R_i be a system of representatives of $(\gamma_i C_H(h_i) \gamma_i^{-1}) \backslash C_G(g)$, which is finite because

$$\begin{aligned}(C_G(g) : \gamma_i C_H(h_i) \gamma_i^{-1}) &= (\gamma_i^{-1} C_G(g) \gamma_i : C_H(h_i)) \\ &= (C_G(h_i) : C_H(h_i)) \ = \ (C_G(h_i) : C_G(h_i) \cap H)\end{aligned}$$

and $(C_G(h_i) : C_G(h_i) \cap H)$ is finite because the index $(G : H)$ is finite. Define the homomorphism

$$\bigoplus_{i=1}^{n} K(X^{h_i})^{C_H(h_i)} \longrightarrow K(X^g)^{C_G(g)},$$

$$(\ E^i\ )_{[h_i]} \quad \longmapsto \quad \bigoplus_{i=1}^{n} F^i \ = \ \bigoplus_{i=1}^{n} \bigoplus_{r\in R_i} (\gamma_i^{-1} r)^* E^i.$$

A (not so formal) way to see that this is well defined is taking the fibre, F_x^i, in a point $x \in X^g$ after acting by $a \in C_G(g)$. First, note that with $r \in R_i$, ra is in $C_G(g)$, so $ra = (\gamma_i\, b\, \gamma_i^{-1})r'$ for some $b \in C_H(h_i)$ and $r' \in R_i$. Furthermore, the mapping $r \mapsto r'$ is a permutation in R_i, then we have

$$\begin{aligned} a^* F_x^i \ &= \ F_{ax}^i = \bigoplus_{r\in R_i} (\gamma_i^{-1} r)^* E_{ax}^i = \bigoplus_{r\in R_i} E^i_{\gamma_i^{-1} rax} = \bigoplus_{r'\in R_i} E^i_{b\gamma_i^{-1} r' x} \\ &= \bigoplus_{r'\in R_i} E^i_{\gamma_i^{-1} r' x} = \bigoplus_{r'\in R_i} (\gamma_i^{-1} r')^* E_x^i = F_x^i\,; \end{aligned}$$

the action of b is removed since E^i is invariant by the action of $C_H(h_i)$.

As before, we obtain a map

$$\text{cores} : \bigoplus_{[h] \text{ in } H} K^*(X^h)^{C_H(h)} \otimes \mathbb{C} \longrightarrow \bigoplus_{[g] \text{ in } G} K^*(X^g)^{C_G(g)} \otimes \mathbb{C}.$$

Now, our setting will be the following: Let $G = \mathrm{PGL}_2(\mathbb{C})$, let Γ be some subgroup of G and let g be an element of the commensurator of Γ, then we define $H = \Gamma \cap g^{-1}\Gamma g$ and $X = \underline{E}\Gamma$, so the Hecke operator

$$T_g : K_\Gamma^*(X) \longrightarrow K_\Gamma^*(X)$$

will be given as the composition

$$K_\Gamma^*(X) \xrightarrow{\text{res}} K_H^*(X) \xrightarrow{\mathrm{Ad}_g} K_{gHg^{-1}}^*(X) \xrightarrow{\text{cores}} K_\Gamma^*(X),$$

where we include the natural map Ad_g given by conjugation. In fact, we defined the operator in $K_G^*(X) \otimes \mathbb{C}$, but in our computations we will be able to drop the $\mathbb{C}$.

Part II
Bianchi groups

Let d be a positive square-free integer, and let $\mathbb{O}_d$ be the ring of integers of the imaginary quadratic extension $\mathbb{Q}(\sqrt{-d})$, then the **Bianchi group** associated to d is defined as

$$\Gamma_d = \mathrm{PSL}_2(\mathbb{O}_d) = \mathrm{SL}_2(\mathbb{O}_d)/\{\pm I\}.$$

In general, except from $d = 3$, these groups can be expressed as amalgamated products, but not all the factor groups are described easily.

This part contains, first, the definition of amalgamated products of groups and HNN extensions, which will be useful to describe some Bianchi groups, then we explain the general algebraic structure of the Bianchi groups, including the amalgam decomposition of the Euclidean Bianchi groups. Later, as the core of the thesis in terms of computations, we develop a study of the group Γ_1, its group cohomology, K-theory and a Hecke operator T_g associated to a prime in $\mathbb{O}_1 = \mathbb{Z}[i]$.

Some of the initial work on Bianchi groups and the group cohomology of Γ_1 are part of my Bachelor's thesis [23].

7 Amalgamated product of groups

Consider a collection of groups $\{G_i\}_{i\in I}$ together with a set $F_{i,j}$ of homomorphisms $G_i \to G_j$, for each pair $i, j \in I$.

Proposition 7.1. *There exists a group G and a collection of homomorphisms $\{f_i\}_{i\in I}$, with $f_i : G_i \to G$ and $f_j \circ f = f_i$, $\forall f \in F_{i,j}$, such that the following property is satisfied:*

- *If there is a group H and a collection of homomorphisms $\{h_i\}_{i\in I}$, $h_i : G_i \to H$, such that $h_j \circ f = h_i$, $\forall f \in F_{i,j}$, then there exists a unique homomorphism $h : G \to H$ that satisfy $h \circ f_i = h_i$, $\forall i \in I$.*

Furthermore, G and $\{f_i\}_{i\in I}$ are unique up to unique isomorphism.

$$\begin{tikzcd}
G_i \arrow[dr, "f_i"] \arrow[drr, bend left, "h_i"] \arrow[dd, "f"'] & & \\
& G \arrow[r, dashed, "h"] & H \\
G_j \arrow[ur, "f_j"'] \arrow[urr, bend right, "h_j"'] & &
\end{tikzcd}$$

Proof. For the existence, take a set of generators S_i of each G_i, then take the disjoint union $\sqcup_{i\in I} S_i$ as the set of generators for G. The relations will be the disjoint union of the relations for each G_i together with $xy^{-1} = e$, whenever $f(x) = y$ for some $f \in F_{i,j}$, with $x \in G_i$ and $y \in G_j$. The f_i are just the inclusions.

The uniqueness is proved using the universal property: Suppose $G, \{f_i\}_{i\in I}$ and $G', \{f'_i\}_{i\in I}$ are such groups, then two homomorphisms $f' : G \to G'$ and $f : G' \to G$ are obtained; the compositions $f \circ f'$ and $f' \circ f$ must be the identity maps on G and G' respectively, so f and f' are isomorphisms. $\square$

G is called the **direct limit** of the G_i relative to the $F_{i,j}$.

Now consider the case where there is a group A and a collection of groups $\{G_i\}_{i\in I}$ with a collection of injective homomorphisms $\{\alpha_i : A \to G_i\}_{i\in I}$, so A is identified with a subgroup of each G_i. The group obtained as the direct limit of $\{A\} \cup \{G_i\}_{i\in I}$ together with the given homomorphisms $\{\alpha_i\}_{i\in I}$ is denoted as $*_A G_i$ and is called the **product of the G_i with A amalgamated**.

Also, in the case of three groups A, G_1, G_2, the **amalgam** is denoted as $G_1 *_A G_2$ and we obtain the respective **amalgamation diagram** as shown below.

$$\begin{tikzcd}
A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & G_1 \arrow[d] \\
G_2 \arrow[r] & G_1 *_A G_2
\end{tikzcd}$$

It can be proved that we can write any $g \in G$ uniquely as a **word** of, first, an element in A and then n interleaved elements of a set of representatives of G_1 and G_2 modulo A, for $n \geq 0$.

Furthermore, we have the presentation

$$G_1 *_A G_2 = \langle\, G_1,\ G_2 \mid \text{ Relations of } G_1, \ \text{ Relations of } G_2, \ \ \alpha_1(a) = \alpha_2(a),\ \forall a \in A \rangle.$$

8 HNN extensions

An HNN extension is a construction similar to that of an amalgamated product.

Let G be a group with a presentation. Suppose there is a collection $\{A_i\}_{i \in I}$ of subgroups of G together with a collection of injections $\{\varphi_i : A_i \to G\}_{i \in I}$. Then the **HNN extension** of G associated to the $\{A_i, \varphi_i\}$ is defined as the group with the presentation

$$G^* = \langle \quad G,\ \{t_i\}_{i \in I} \quad \mid \quad \text{Relations of } G,\ t_i a t_i^{-1} = \varphi_i(a),\ \text{ for } i \in I,\ a \in A_i \quad \rangle.$$

G is called the **base**, $\{t_i\}_{i \in I}$ is called the **free part**, and the $\{A_i, \varphi_i(A_i)\}_{i \in I}$ are called the **associated subgroups**. The size of the set I is the **free part rank**.

Consequently, a group is called an **HNN group** if it is the HNN extension of some group with some associated subgroups.

We will only use HNN extensions associated to one subgroup and its inclusion, in this case, with $A \subset G$, we have

$$G^* = \langle \quad G,\ t \quad \mid \quad \text{Relations of } G,\ tat^{-1} = a,\ \text{ for } a \in A \quad \rangle.$$

As in amalgamated products, there is a way to write uniquely each element of a HNN extension. This is known as the Britton's lemma.

Using the notation used in the initial definition, let S_i be a set of representatives of G modulo A_i and let R_i be a set of representatives of G modulo $\varphi_i(A_i)$. Then every $g \in G^*$ is written uniquely as

$$g_0\, t_{i_1}^{e_1}\, g_1\, t_{i_2}^{e_2} \cdots t_{i_k}^{e_k}\, g_k, \quad \text{with } e_j = \pm 1,$$

where $g_o \in G$, while $g_j \in S_{t_j}$ if $e_j = -1$ and $g_j \in R_{t_j}$ if $e_j = 1$; also, there is no subsequence $t^e \cdot 1 \cdot t^{-e}$.

An example of this is the group with the presentation

$$\langle s, t, u \mid s^2 = (st)^3 = [t, u] = 1 \rangle.$$

This group is both the amalgamated product $\mathrm{PSL}_2(\mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})$ and the HNN extension of $\mathrm{PSL}_2(\mathbb{Z}) = \langle s, t \mid s^2 = (st)^3 = 1 \rangle$ associated to the subgroup generated by t (together with the inclusion).

9 Bianchi groups

As said before, the Bianchi group associated with a positive square-free integer d is defined as

$$\Gamma_d = \mathrm{PSL}_2(\mathbb{O}_d)$$

where $\mathbb{O}_d$ is the ring of integers of the imaginary quadratic extension $\mathbb{Q}(\sqrt{-d})$.

We can describe these rings explicitly: With $\delta = \sqrt{-d}$ and $\eta = \frac{1}{2}(1 + \delta)$, we have

$$\mathbb{O}_d = \mathbb{Z}[\delta] \quad \text{for} \quad d \equiv 1, 2 \mod 4, \quad \text{and}$$

$$\mathbb{O}_d = \mathbb{Z}[\eta] \quad \text{for} \quad d \equiv 3 \mod 4.$$

This is shown easily. See [1, Chapter 13].

9.1 Subgroup of elementary matrices

Let R be a ring. Let $x \in R$ be any element and $\mu \in R$ a unit. We define the matrices

$$E(x) = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad D(\mu) = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$

The matrices $E(x)$ are called **elementary matrices**, and the group generated by them, $E_2(R)$, is called the 2×2 **elementary matrix group**.

A theorem of P. M. Cohn [5] provides a presentation of the group $E_2(R)$ for certain subrings of $\mathbb{C}$; it is as follows.

Theorem 9.1. *Let R be a subring of $\mathbb{C}$ with the usual absolute value, such that the range of values is well-ordered and if $\alpha \in R$ with $|\alpha|^2 < 4$ then $|\alpha|^2$ is an integer. Then, a presentation for $E_2(R)$ is given by the generators $E(x)$, $x \in R$, and the relations*

- $E(x)E(0)E(y) = -E(x+y), \quad x, y \in R$;
- $E(x)D(\mu) = D(\mu^{-1})E(\mu x \mu), \quad x, \mu \in R$, μ *unit;*
- $(E(\alpha)E(\bar{\alpha}))^p = -I, \quad$ *for all* $\alpha \in R$ *with* $|\alpha| = \sqrt{p}, \quad p \in \{2, 3\}$;
- $E(\mu)E(\mu^{-1})E(\mu) = -D(\mu), \quad \mu \in R$ *unit.*

From the first relation we can deduce that $E(0)^2 = -I$, having $x = 0$ and taking out the $E(y)$. Hence $E(0)^{-1} = -E(0)$.

Now, take ξ such that $\mathbb{O}_d = \mathbb{Z}[\xi]$. The rings $\mathbb{Z}[\xi]$ satisfy the hypotheses.

Let $x = a + b\xi$, with a and b positive integers. We see from the first relation that $E(x) = E(a)E(0)^{-1}E(b\xi)$, and then

$$\begin{aligned} E(x) &= E(a-1)E(0)^{-1}E(1) \cdot E(0)^{-1} \cdot E((b-1)\xi)E(0)^{-1}E(\xi) \\ &= \underbrace{E(1)E(0)^{-1}E(1) \ldots E(1)E(0)^{-1}E(1)}_{a \text{ times}} E(0)^{-1} \underbrace{E(\xi)E(0)^{-1}E(\xi) \ldots E(\xi)E(0)^{-1}E(\xi)}_{b \text{ times}}. \end{aligned}$$

Also, if a or b are negative we could use -1 or $-\xi$ respectively. So, we can reduce the set of generators to $\{E(0), E(1), E(-1), E(\xi), E(-\xi)\}$. We will see later that also $E(-1)$ and $E(-\xi)$ are not necessary.

9.2 The Euclidean Bianchi groups and their amalgam decompositions

The rings $\mathbb{O}_d$ are an Euclidean domain only when

$$d = 1, 2, 3, 7, 11.$$

For a proof see [12, Theorem 246]. For that reason, these Γ_d are called the **Euclidean Bianchi groups**.

Cohn [6, Theorems 6.1 and 9.3 and further discussions] states that for the Euclidean cases we have the equation

$$E_2(\mathbb{O}_d) = \mathrm{SL}_2(\mathbb{O}_d).$$

In this way we will obtain finite presentations for the Euclidean Bianchi groups and then deduce their amalgam decomposition. The presentations for Γ_1 and Γ_7 are explained in [9]; here we will develop completely the presentation for the group Γ_2.

The units in $\mathbb{O}_2$ are just 1 and -1. Let $\delta = \sqrt{-2} = i\sqrt{2}$. So, for $E_2(\mathbb{O}_2) = \mathrm{SL}_2(\mathbb{O}_2)$, we have that a complete set of relations in terms of the generators $E(x)$, $x \in \mathbb{O}_2$, and J is

(1) $E(x)E(0)E(y) = JE(x+y), \quad x, y \in \mathbb{O}_2;$

(2) $J^2 = I, \quad J$ central;

(3) $(E(\delta)E(-\delta))^2 = (E(1+\delta)E(1-\delta))^3 = J;$

(4) $E(1)^3 = J, \quad E(-1)^3 = I.$

As seen before, we can reduce the set of generators to $E(0)$, $E(1)$, $E(-1)$, $E(\delta)$, $E(-\delta)$, and J. In this way, we would have by *definition* that, for a positive integer n, $E(n)$ is written in terms of the generators as $E(1)E(0)^{-1}E(1)\ldots E(1)E(0)^{-1}E(1)$ (n times); the same for $E(-n), E(n\delta)$, and $E(-n\delta)$, and therefore for any $E(a+b\delta)$ written as $E(a)E(0)^{-1}E(b\delta)$.

Now, we claim that the generators $E(0), E(1), E(-1), E(\delta), E(-\delta)$, and J together with the relations

(a) $E(0)^2 = E(1)^3 = J;$

(b) $J^2 = I, \quad J$ central;

(c) $(E(\delta)E(-\delta))^2 = (E(1+\delta)E(1-\delta))^3 = J;$

(d) $E(1)E(0)E(\delta) = E(\delta)E(0)E(1),$
$E(0)E(1)E(0)E(-1) = E(0)E(\delta)E(0)E(-\delta) = I;$

(e) $E(1+\delta) = E(1)E(0)^{-1}E(\delta), \quad E(1-\delta) = E(1)E(0)^{-1}E(-\delta);$

are equivalent to the previous presentation. Clearly the first presentation implies this one, so we see the other direction.

Since we already know what does any $E(x)$ mean in terms of the new generators, the relations (1) are obtained simply from the pseudo-commuting relation in (d), which can be written as $E(1)E(0)^{-1}E(\delta) = E(\delta)E(0)^{-1}E(1)$ (because $E(0) = JE(0)^{-1}$).

Indeed, if $x = a + b\delta$ and $y = c + d\delta$, with a, b, c, d positive, then we could take $E(x)E(0)^{-1}E(y)$ and move all the $E(1)$'s to the left and all the $E(\delta)$'s to the right to obtain $E(x+y) = E((a+c)+(b+d)\delta)$. For the cases where a, b, c, or d are negative, we should use the relations

$$
\begin{gathered}
E(1)E(0)^{-1}E(-\delta) = E(-\delta)E(0)^{-1}E(1), \\
E(-1)E(0)^{-1}E(\delta) = E(\delta)E(0)^{-1}E(-1), \\
E(-1)E(0)^{-1}E(-\delta) = E(-\delta)E(0)^{-1}E(-1), \\
E(1)E(0)^{-1}E(-1) = E(-1)E(0)^{-1}E(1) = E(0), \\
E(\delta)E(0)^{-1}E(-\delta) = E(-\delta)E(0)^{-1}E(\delta) = E(0),
\end{gathered}
$$

which can be deduced from (d).

The only relation left to verify is $E(-1)^3 = I$, but this is obtained easily from (a) and the equation $E(-1) = E(0)^{-1}E(1)^{-1}E(0)^{-1}$, which comes from (d).

Now, we should make another reduction to the presentation for $E_2(\mathbb{O}_2)$. First, we see that (e) is unnecessary since we can remove it and replace that in (c); besides, (d) gives an expression for $E(-1)$ and $E(-\delta)$ in terms of $E(0), E(1)$, and $E(\delta)$, so we can remove both generators and replace the expressions where it is necessary.

Define

$$A = E(0)^{-1}, \quad T = E(0)E(1)^{-1}, \quad \text{and} \quad U = E(0)E(\delta)^{-1}.$$

With the generators A, T, U, and J, the last relations would be

(a) $(A^{-1})^2 = (T^{-1}A^{-1})^3 = J$;

(b) $J^2 = I, \quad J$ central;

(c) $(U^{-1}AUA)^2 = (T^{-1}U^{-1}A^{-1}T^{-1}AAUA)^3 = J$;

(d) $T^{-1}A^{-1}A^{-1}U^{-1}A^{-1} = U^{-1}A^{-1}A^{-1}T^{-1}A^{-1}$.

This is equivalent to the presentation

$$\langle A, T, U, J \mid J^2 = I, J \text{ central}, \;\; A^2 = (AT)^3 = (U^{-1}AUA)^2 = J, \;\; [T, U] = I \rangle.$$

The second relation in (c) is omitted because it can be obtained from the others. Indeed, we have, using $A^{-1} = JA,\; T^{-1}A^{-1}T^{-1} = ATA$, and other relations,

$$
\begin{aligned}
&(T^{-1}U^{-1}A^{-1}T^{-1}AAUA)^3 \\
&\qquad = (T^{-1}U^{-1}A^{-1}T^{-1}UA^{-1})^3 = (T^{-1}U^{-1}A^{-1}U(T^{-1}A^{-1}T^{-1})T)^3 \\
&\qquad = (T^{-1}(U^{-1}A^{-1}UA)TAT)^3 = (JT^{-1}A^{-1}U^{-1}AUTAT)^3 \\
&\qquad = J^3T^{-1}A^{-1}U^{-1}AUT(ATT^{-1}A^{-1})U^{-1}AUT(ATT^{-1}A^{-1})U^{-1}AUTAT \\
&\qquad = JT^{-1}A^{-1}U^{-1}A(UTU^{-1})A(UTU^{-1})AUTAT \\
&\qquad = JT^{-1}A^{-1}U^{-1}(ATATAT)UAT \\
&\qquad = J^2T^{-1}A^{-1}U^{-1}UAT = I.
\end{aligned}
$$

Finally, adding the relation $J = I$, we obtain what must be the simplest presentation for this Bianchi group:

$$\Gamma_2 = \mathrm{PSL}_2(\mathbb{O}_2) = \langle a, t, u \mid a^2 = (at)^3 = (u^{-1}aua)^2 = [t, u] = 1\rangle.$$

Now, to deduce the amalgam decomposition, we have to do some changes to the last presentation.

Take $s = at$, $v = u^{-1}su$, and $m = u^{-1}au$. We may obtain

$$
\begin{aligned}
\langle \ a, m, s, u, v \ \mid \ & a^2 = m^2 = s^3 = v^3 = (am)^2 = (sv^{-1})^2 = 1, \\
& am = sv^{-1},\ m = u^{-1}au,\ v = u^{-1}su \ \rangle.
\end{aligned}
$$

With this, define

$$G_1 = \langle a, m, u \mid a^2 = m^2 = (am)^2 = 1,\ m = u^{-1}au\rangle,$$

and

$$G_2 = \langle s, v, u \mid s^3 = v^3 = (sv^{-1})^2 = 1,\ v = u^{-1}su\rangle,$$

so Γ_2 is the free product of G_1 and G_2 with the identifications $u = u$ and $am = sv^{-1}$.

Note that G_1 is the HNN extension of the Klein group $C_2 \times C_2$ with the associated subgroups $\langle a = (1, 0)\rangle$ and $\langle m = (0, 1)\rangle$ (and the monomorphism $a \mapsto m$). And, G_2 is an HNN extension of the alternating group A_4 with the associated subgroups $\langle s = (123)\rangle$ and $\langle v = (134)\rangle$ (s and v could be any pair of generators for A_4 such that sv^{-1} is a product of two transpositions; G_2 will be the same).

We can see there is a common subgroup of G_1 and G_2, this is the group $\mathbb{Z} * C_2$. In G_1, it is the subgroup $\langle u\rangle * \langle am\rangle$, and in G_2 it is $\langle u\rangle * \langle sv^{-1}\rangle$. (The equalities $\langle u, am\rangle = \langle u\rangle * \langle am\rangle$ and $\langle u, sv^{-1}\rangle = \langle u\rangle * \langle sv^{-1}\rangle$ can be verified with the presentations.)

From all the above we can conclude the amalgam structure of Γ_2.

Proposition 9.2. *We have*

$$\Gamma_2 \cong G_1 *_{(\mathbb{Z} * C_2)} G_2,$$

where G_1 is the HNN extension of $C_2 \times C_2$ associating two generators and G_2 is the HNN extension of A_4 associating two 3-cycles.

For the rest of the Euclidean Bianchi groups, except Γ_3, we have the following amalgam decompositions:

$$\Gamma_1 = \big(A_4 *_{C_3} S_3\big) *_{\mathrm{PSL}_2(\mathbb{Z})} \big(S_3 *_{C_2} D_2\big);$$

$$\Gamma_7 = \big(\mathbb{Z} * C_2\big) *_{(\mathbb{Z} * C_2 * C_2)} G,$$

where G is the HNN extension of $S_3 *_{C_2} S_3$ associating a 3-cycle with itself; and

$$\Gamma_{11} = \big(\mathbb{Z} * C_3\big) *_{(\mathbb{Z} * C_3 * C_3)} G,$$

where G is the HNN extension of $A_4 *_{C_3} A_4$ associating a 3-cycle with itself. These can be proved exactly the same way. See [9] for further information.

Since the only group that does not contain an HNN extension as factor group is Γ_1, we will be able to compute its cohomology groups *directly* using a Mayer–Vietoris spectral sequence associated to its classifying space.

9.3 The non-Euclidean Bianchi groups

Theorem 9.3. *For any $d \neq 1, 2, 3, 7, 11$, we have the presentation*

$$PE_2(\mathbb{O}_d) = \langle\, a, t, u \mid a^2 = (at)^3 = [t, u] = 1 \,\rangle.$$

Proof. The rings $\mathbb{O}_d$ have no units apart from ± 1, and there are no elements α such that $|\alpha| < 2$. So from the theorem of Cohn, $E_2(\mathbb{O}_d)$ has the generators J and $E(x)$, for $x \in \mathbb{O}_d$, with the relations

- $E(x)E(0)E(y) = JE(x+y), \quad x, y \in R;$
- $J^2 = I, \quad J$ central;
- $E(1)^3 = J, \quad E(-1)^3 = I.$

As we did with Γ_2, we may reduce to

$$E_2(\mathbb{O}_d) = \langle\, J, E(0), E(1), E(\xi) \mid E(0)^2 = E(1)^3 = J,\ J \text{ central},\ J^2 = I,$$
$$E(1)E(0)E(\xi) = E(\xi)E(0)E(1)\,\rangle.$$

Then, letting $A = E(0)^{-1},\ T = E(0)E(1)^{-1},\ U = E(0)E(\xi)^{-1}$ we have

$$E_2(\mathbb{O}_d) = \langle\, J, A, T, U \mid A^2 = (AT)^3 = J,\ J \text{ central},\ J^2 = [T, U] = I\,\rangle.$$

Identifying a, t, u with A, T, U after making $J = I$ the result is obtained. $\square$

Note that we have already seen this group; this was the example mentioned as a group that is both an amalgamated product and an HNN extension. So we have the isomorphism

$$PE_2(\mathbb{O}_d) \cong \mathrm{PSL}_2(\mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})$$

and the fact that $PE_2(\mathbb{O}_d)$ is also the HNN extension of $\mathrm{PSL}_2(\mathbb{Z})$ by an infinite cyclic subgroup.

Furthermore, B. Fine [9] exhibited an amalgam decomposition for all the non-Euclidean Bianchi groups as

$$\Gamma_d \cong PE_2(\mathbb{O}_d) *_H G_d$$

where H is an amalgam of two copies of $\mathrm{PSL}_2(\mathbb{Z})$ and G_d is a particular group depending on d. This is proved with Poincaré polygons and polyhedrons, using that the Γ_d act on the non-Euclidean hyperbolic 3-space, where the action defines some particular regions (polygons/polyhedrons) that lead to a construction of presentations for the Γ_d.

Later, we will also use this action to illustrate the classifying space for proper actions for the group Γ_1.

10 The group Γ_1

From the previous section we have the isomorphism

$$\Gamma_1 \cong \big(A_4 *_{C_3} S_3\big) *_{\mathrm{PSL}_2(\mathbb{Z})} \big(S_3' *_{C_2} D_2\big),$$

with $\mathrm{PSL}_2(\mathbb{Z}) = C_3' * C_2'$ and the intersections $A_4 \cap S_3' = C_3'$, $A_4 \cap D_2 = \{1\}$, $S_3 \cap S_3' = \{1\}$, and $S_3 \cap D_2 = C_2'$. These intersections can be seen easily from the presentation

$$\Gamma_1 = \langle\, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \mid \mathbf{a}^3 = \mathbf{b}^2 = \mathbf{c}^3 = \mathbf{d}^2 = (\mathbf{ac})^2 = (\mathbf{ad})^2 = (\mathbf{bd})^2 = (\mathbf{bc})^2 = 1 \,\rangle$$

that is equivalent to the amalgam decomposition. Here $A_4 = \langle \mathbf{a}, \mathbf{c} \rangle$, $S_3 = \langle \mathbf{a}, \mathbf{d} \rangle$, $D_2 = \langle \mathbf{b}, \mathbf{d} \rangle$, and $S_3' = \langle \mathbf{b}, \mathbf{c} \rangle$, so $C_3 = \langle \mathbf{a} \rangle$, $C_2 = \langle \mathbf{b} \rangle$, $C_3' = \langle \mathbf{c} \rangle$, and $C_2' = \langle \mathbf{d} \rangle$.

We can give explicit matrices that represent the generators, namely

$$\mathbf{a} = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

10.1 Group cohomology

The method used in this section can be applied to the other Bianchi groups with amalgam decompositions, but the case of Γ_1 is the one in which we need only group cohomology of finite groups.

These computations are part of my previous work in [23].

Let X be a model for $B\Gamma_1$, the classifying space for Γ_1, and let $X_{11} = BA_4$, $X_{12} = BS_3$, $X_{21} = BS_3'$, $X_{22} = BD_2$, $Y_1 = BC_3$, $Y_2 = BC_2$, and $Z = B\mathrm{PSL}_2(\mathbb{Z})$, so we have

$$X \cong \big(X_{11} \cup_{Y_1} X_{12}\big) \cup_Z \big(X_{21} \cup_{Y_2} X_{22}\big).$$

With this, we obtain a covering $\{X_{11}, X_{12}, X_{21}, X_{22}\}$ for a classifyng space of Γ_1. We can use this to construct a Mayer–Vietoris spectral sequence (see Section 2.2) whose E_1-term is given by

$$\begin{aligned} E_1^{0,q} &= H^q(X_{11}) \oplus H^q(X_{12}) \oplus H^q(X_{21}) \oplus H^q(X_{22}) \\ &\cong H^q(A_4) \oplus H^q(S_3) \oplus H^q(S_3') \oplus H^q(D_2), \end{aligned}$$

$$\begin{aligned} E_1^{1,q} &= H^q(X_{11} \cap X_{12}) \oplus H^q(X_{11} \cap X_{21}) \oplus H^q(X_{12} \cap X_{22}) \oplus H^q(X_{21} \cap X_{22}) \\ &\cong H^q(C_3) \oplus H^q(C_3') \oplus H^q(C_2') \oplus H^q(C_2), \end{aligned}$$

for $q \geq 0$, and $E_1^{p,q}$ trivial for $p \geq 2$, where the differentials of bidegree $(1,0)$ are all induced by inclusions. This spectral sequence converges to $H^*(B\Gamma_1) = H^*(\Gamma_1)$.

For the cyclic group C_n of order n, the symmetric group S_3, and the Klein group D_2, it is known that

$$H^k(C_n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0, \\ 0, & k \text{ odd}, \\ \mathbb{Z}/n\,\mathbb{Z}, & k>0 \text{ even}; \end{cases} \qquad H^k(S_3;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0, \\ 0, & k \text{ odd}, \\ \mathbb{Z}/2\,\mathbb{Z}, & k \equiv 2 \bmod 4, \\ \mathbb{Z}/6\,\mathbb{Z}, & k \equiv 0 \bmod 4,\ k>0; \end{cases}$$

$$\text{and} \qquad H^k(D_2;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0, \\ (\mathbb{Z}/2\,\mathbb{Z})^{(k-1)/2}, & k \text{ odd}, \\ (\mathbb{Z}/2\,\mathbb{Z})^{(k+2)/2}, & k \text{ even},\ k>0. \end{cases}$$

The later can be proved with the Künneth formula (see [13] for the definition). And, using GAP [11], the first cohomology groups of A_4 are

$$H^0(A_4)=\mathbb{Z}, \quad H^1(A_4)=0, \quad H^2(A_4)=C_3,$$

$$H^3(A_4)=C_2, \quad H^4(A_4)=C_6, \quad H^5(A_4)=0.$$

Then the E_1-term looks like this:

$$\begin{array}{c|ccccc}
 & \vdots & & \vdots & & \\
5 & C_2^4 & \longrightarrow & 0 & \longrightarrow & 0 \ \cdots \\
4 & C_6^3 \oplus C_2^3 & \xrightarrow{d_1^{0,4}} & C_3^2 \oplus C_2^2 & \longrightarrow & 0 \ \cdots \\
3 & C_2 \oplus C_2 & \longrightarrow & 0 & \longrightarrow & 0 \ \cdots \\
2 & C_3 \oplus C_2^4 & \xrightarrow{d_1^{0,2}} & C_3^2 \oplus C_2^2 & \longrightarrow & 0 \ \cdots \\
1 & 0 & \longrightarrow & 0 & \longrightarrow & 0 \ \cdots \\
0 & \mathbb{Z}^4 & \xrightarrow{d_1^{0,0}} & \mathbb{Z}^4 & \longrightarrow & 0 \ \cdots \\
\hline
 & 0 & & 1 & & 2
\end{array}$$

With all the corresponding inclusions, we define the group homomorphims

$$\begin{aligned}
\alpha_1 : H^*(A_4) \to H^*(C_3) \qquad &\text{and} \qquad \alpha_2 : H^*(A_4) \to H^*(C_3'),\\
\beta_1 : H^*(S_3) \to H^*(C_3) \qquad &\text{and} \qquad \beta_2 : H^*(S_3) \to H^*(C_2'),\\
\boldsymbol{\gamma}_1 : H^*(S_3') \to H^*(C_3') \qquad &\text{and} \qquad \boldsymbol{\gamma}_2 : H^*(S_3') \to H^*(C_2),\\
\delta_1 : H^*(D_2) \to H^*(C_2') \qquad &\text{and} \qquad \delta_2 : H^*(D_2) \to H^*(C_2),
\end{aligned}$$

induced in cohomology, so we have

$$d_1^{0,q} = \alpha_1 + \beta_1 \oplus \alpha_2 + \boldsymbol{\gamma}_1 \oplus \beta_2 + \delta_1 \oplus \boldsymbol{\gamma}_2 + \delta_2 \ : \ E_1^{0,q} \longrightarrow E_1^{1,q}.$$

The $\oplus$ are used to separate the components in each direct sum; the $+$ denote the (abelian) operation in each component. You can go back to the direct sum decomposition of $E_1^{0,q}$ and $E_1^{1,q}$ to make this clear.

We deal with these morphisms by separated cases in order to confirm that the induced homomoprhisms are not trivial as long as they can be not trivial. Since all the homomorphisms listed go to cyclic groups, the non-triviality will leave just one other option (where the final results do not change).

Using the previous notation for $A_4 = \langle \mathbf{a}, \mathbf{c} \rangle$, note that there is a group homomorphism $j : A_4 \to C_3 = \langle \mathbf{a} \rangle$ given by $j(\mathbf{a}) = \mathbf{a}$ and $j(\mathbf{c}) = \mathbf{a}^2$. Then, the composition $j \circ i$ with the inclusion into A_4 is the identity map on C_3; this implies that the induced morphism

$$(j \circ i)^* = i^* \circ j^* : H^*(C_3) \to H^*(C_3)$$

is the identity map as well, which means that $i^* = \alpha_1$ must be not trivial whenever $H^*(C_3)$ is not trivial. The same is obtained for α_2.

For the group S_3, the fact that it is an extension of C_3 by C_2 gives a homomorphism $S_3 \to C_2$ which becomes the identity on C_2 when composed with the inclusion. Using the same argument we obtain that β_2 and $\boldsymbol{\gamma}_2$ are trivial only when they *must* be trivial.

For β_1 and $\boldsymbol{\gamma}_1$, this is not immediate. The result comes from how the cohomology of S_3 can be computed from a Lyndon-Hochschild-Serre spectral sequence using the isomorphism $S_3 \cong C_3 \rtimes C_2$. This is made explicitly in [23], where we conclude that these morphisms are non-trivial in even cohomology groups.

At last, the group D_2 is isomorphic to the direct product $C_2 \oplus C_2$, hence there is a homomorphism $D_2 \to C_2$ (the projection) that, composed with the inclusion, is equal to the identity map on C_2. As before, δ_1 and δ_2 are then not trivial whenever they are not forced to be trivial.

Now we can give explicitly the differentials in the E_1-term. For $d_1^{0,0}$, all the homomorphisms are identity maps between $\mathbb{Z}$'s, then

$$d_1^{0,0} : (a, b, c, d) \mapsto (a + b, a + c, b + d, c + d),$$

and we have

$$\mathrm{Ker}(d_1^{0,0}) \cong \mathbb{Z}, \qquad \mathrm{Im}(d_1^{0,0}) \cong \mathbb{Z}^3, \qquad \text{and} \qquad E_1^{1,0}/\mathrm{Im}(d_1^{0,0}) \cong \mathbb{Z}\,.$$

For $d_1^{0,2}$,

$$d_1^{0,2} : (a, b, c, d_1, d_2) \mapsto (a, a, b + d_1, c + d_1),$$

(in the image, d_1 may be replaced by d_2 or $d_1 + d_2$; the result is the same) then

$$\mathrm{Ker}(d_1^{0,2}) \cong C_2 \oplus C_2, \qquad \mathrm{Im}(d_1^{0,2}) \cong C_3 \oplus C_2 \oplus C_2, \qquad \text{and} \qquad E_1^{1,2}/\mathrm{Im}(d_1^{0,2}) \cong C_3.$$

For $d_1^{0,4}$,

$$d_1^{0,4} : (a, b, c, d_1, d_2, d_3) \mapsto (a_{(3)} + b_{(3)}, a_{(3)} + c_{(3)}, b_{(2)} + d_1, c_{(2)} + d_1),$$

(in the image, d_1 may be replaced by any other nontrivial sum of d_i's; the result is the same). Doing the computations we obtain

$$\mathrm{Ker}(d_1^{0,4}) \cong C_6 \oplus C_2^3, \qquad \mathrm{Im}(d_1^{0,4}) = E_1^{1,4}, \qquad \text{and} \qquad E_1^{1,4}/\mathrm{Im}(d_1^{0,4}) = 0.$$

Then E_2 looks like this:

$$
\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
5 & C_2^4 & 0 & 0 & \cdots \\
4 & C_6 \oplus C_2^3 & 0 & 0 & \cdots \\
3 & C_2 \oplus C_2 & 0 & 0 & \cdots \\
2 & C_2 \oplus C_2 & C_3 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} & \mathbb{Z} & 0 & \cdots \\
\hline
 & 0 & 1 & 2 &
\end{array}
$$

From the comments made in Section 2.1, since the fourth diagonal is the only one where there is more than one non-trivial factor, we have the group extension

$$0 \longrightarrow C_3 \longrightarrow H^4(\Gamma_1) \longrightarrow C_2 \oplus C_2 \longrightarrow 0.$$

But the only abelian extension of these groups is their direct product.

Finally, we got the first six cohomology groups for Γ_1:

$$H^0(\Gamma_1) = \mathbb{Z}, \qquad H^1(\Gamma_1) = \mathbb{Z},$$

$$H^2(\Gamma_1) = C_2^2, \qquad H^3(\Gamma_1) = C_2^2 \oplus C_3,$$

$$H^4(\Gamma_1) = C_2^4 \oplus C_3, \qquad H^5(\Gamma_1) = C_2^4.$$

The only non-periodic cohomology is that of A_4, so we can compute the cohomology of Γ_1 as far as we can compute the cohomology of A_4.

More explicitly, we have the E_1-term, for $k > 0$, as

$$
\begin{array}{c|ccccc}
4k+3 & H^*(A_4) \oplus C_2^{2k+1} & \longrightarrow & 0 & \longrightarrow & 0 & \cdots \\
4k+2 & H^*(A_4) \oplus C_2^2 \oplus C_2^{2k+2} & \longrightarrow & C_3^2 \oplus C_2^2 & \longrightarrow & 0 & \cdots \\
4k+1 & H^*(A_4) \oplus C_2^{2k} & \longrightarrow & 0 & \longrightarrow & 0 & \cdots \\
4k & H^*(A_4) \oplus C_6^2 \oplus C_2^{2k+1} & \longrightarrow & C_3^2 \oplus C_2^2 & \longrightarrow & 0 & \cdots \\
\hline
 & 0 & & 1 & & 2 &
\end{array}
$$

The differential $d_1^{0,4k}$ is surjective always due to the surjectivity of β_1 and β_2. Its kernel is isomorphic to $H^{4k}(A_4) \oplus C_2^{2k+1}$; this can be seen by thinking that for each element in $H^{4k}(A_4)$, the options to go to zero are given by a C_2^{2k+1}.

Conversely, for $d_1^{0,4k+2}$, since β_1 and β_2 are zero maps, we have

$$\operatorname{Ker}(d_1^{0,4k+2}) \cong \operatorname{Ker}(\alpha_1) \oplus C_2^{2k+2} \quad \text{and} \quad \operatorname{Im}(d_1^{0,4k+2}) \cong \operatorname{Im}(\alpha_1) \oplus C_2^2.$$

In the quotient $E_1^{1,4k+2}/\mathrm{Im}\ d_1$ the components with C_2 become trivial.

These pairs of kernels and images can be verified using the first isomorphism theorem.

We have then the E_2-term:

$$
\begin{array}{c|ccc}
4k+3 & H^*(A_4) \oplus C_2^{2k+1} & 0 & 0 \\
4k+2 & \operatorname{Ker}(\alpha_1) \oplus C_2^{2k+2} & C_3^2/\operatorname{Im}(\alpha_1 \oplus \alpha_2) & 0 \\
4k+1 & H^*(A_4) \oplus C_2^{2k} & 0 & 0 \\
4k & H^*(A_4) \oplus C_2^{2k+1} & 0 & 0 \\
\hline
 & 0 & 1 & 2
\end{array}
$$

With this we can conclude that, for $q \not\equiv 3 \bmod 4$,

$$H^q(\Gamma_1) = E_\infty^{0,q} = E_2^{0,q}.$$

For $q \equiv 3 \bmod 4$, there is an exact sequence

$$0 \longrightarrow C_3^2/\mathrm{Im}(\widetilde{\alpha}) \longrightarrow H^q(\Gamma_1) \longrightarrow H^q(A_4) \oplus C_2^{(q-1)/2} \longrightarrow 0,$$

where $\widetilde{\alpha}$ is the morphism $H^{q-1}(A_4) \to C_3^2$ given by (the direct sum of) two copies of the induced morphism $H^{q-1}(A_4) \to H^{q-1}(C_3) = C_3$.

10.2 Classifying space for proper actions

Now we give the G-CW-complex structure of a model for $X = \underline{E}\Gamma_1$, which has dimension 2. Let

$$X^{(0)} = \Gamma_1/A_4 \times \{p\} \coprod \Gamma_1/S_3 \times \{q\} \coprod \Gamma_1/D_2 \times \{r\} \coprod \Gamma_1/S_3' \times \{s\},$$

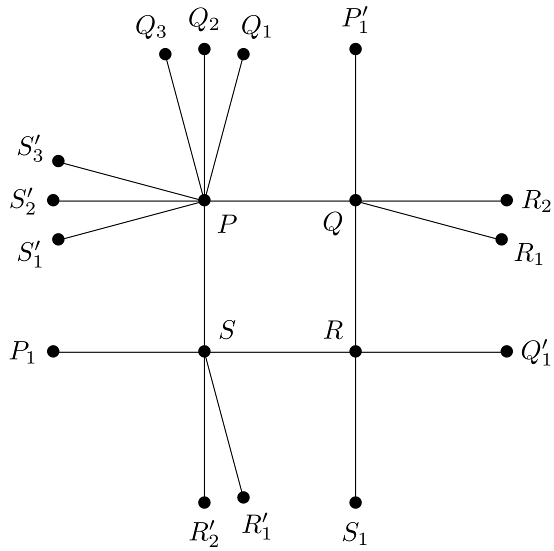
where each $\mathbb{D}^0$ (point) has been labelled with a letter. The 1-skeleton is obtained from the pushout

$$\begin{array}{ccc} \Gamma_1/C_3 \times \mathbb{S}^0 \coprod \Gamma_1/C_2' \times \mathbb{S}^0 \coprod \Gamma_1/C_2 \times \mathbb{S}^0 \coprod \Gamma_1/C_3' \times \mathbb{S}^0 & \xrightarrow{\varphi} & X^{(0)} \\ \Big\downarrow{\scriptstyle \text{inclusion}} & & \Big\downarrow \\ \Gamma_1/C_3 \times \mathbb{D}^1 \coprod \Gamma_1/C_2' \times \mathbb{D}^1 \coprod \Gamma_1/C_2 \times \mathbb{D}^1 \coprod \Gamma_1/C_3' \times \mathbb{D}^1 & \longrightarrow & X^{(1)} \end{array}$$

so that $X^{(1)}$ is the union of $X^{(0)}$ and many copies of $\mathbb{D}^1$, identifying the image by φ and the inclusion, respectively, of many copies of $\mathbb{S}^0$. Writing each copy of $\mathbb{S}^0$ as two ordered points $\{-1, 1\}$ and denoting a point in $X^{(0)}$ just as the coset, the map φ is defined as follows: For any $\gamma \in \Gamma_1$,

$$\begin{aligned} \varphi : \quad \gamma C_3 \times \{-1,1\} &\mapsto \{\gamma A_4 \,,\, \gamma S_3\}, \\ \gamma C_2' \times \{-1,1\} &\mapsto \{\gamma S_3 \,,\, \gamma D_2\}, \\ \gamma C_2 \times \{-1,1\} &\mapsto \{\gamma D_2 \,,\, \gamma S_3'\}, \\ \gamma C_3' \times \{-1,1\} &\mapsto \{\gamma S_3' \,,\, \gamma A_4\}. \end{aligned}$$

This means that we will add a segment between two points whenever their corresponding cosets intersect as a coset of any of the cyclic groups in Γ_1. Take P, Q, R, S as the trivial cosets of A_4, S_3, D_2, S_3', respectively. The space $X^{(1)}$ would begin to look like this:

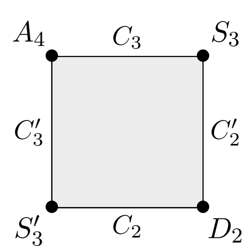


The lines PQ_i, $i = 1, 2, 3$, come from the cosets $\mathbf{c}C_3$, $\mathbf{c}^2C_3$, and $\mathbf{ac}^2C_3$, respectively. There are no more cosets of S_3 connected to A_4. It continues similarly.

Finally, we add a 2-cell, filling the square:

$$\begin{tikzcd}
\Gamma_1/\{1\} \times \mathbb{S}^1 \arrow[r] \arrow[d] & X^{(1)} \arrow[d] \\
\Gamma_1/\{1\} \times \mathbb{D}^2 \arrow[r] & X^{(2)} = X
\end{tikzcd}$$

This space is proper since all the isotropy groups are finite groups, and this is because X can be thought as the space obtained from a square by the action of Γ_1 with the isotropy groups showed below.



Also, X is indeed a model for $\underline{E}\Gamma_1$, since every fixed space X^H, H finite subgroup of Γ_1, is weakly contractible.

An alternate description/construction of this space can be found in [10] and [24].

10.2.1 In the hyperbolic space

It is known that the group $\mathrm{PSL}_2(\mathbb{C})$ acts on the hyperbolic 3-space $\mathbb{H}^3$. We can describe the action thinking of $\mathbb{H}^3$ as quaternions.

Let $\mathbb{H}^3$ be the upper half space $\{(z,\xi) \ : \ \xi > 0\} \subset \mathbb{C} \times \mathbb{R}$, considering each pair as a quaternion:

$$(z,\xi) = (x+iy,\xi) = x + iy + j\xi,$$

where i, j, and ij are the quaternion units. Then, for $q \in \mathbb{H}^3$, define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot q = (aq+b)(cq+d)^{-1},$$

where $q^{-1} = \dfrac{\overline{q}}{||q||}$. It can be shown that the action is well defined, that is, the image of q is contained in $\mathbb{H}^3$.

With this action, we can view the space $\underline{E}\Gamma_1$ described before inside $\mathbb{H}^3$. Let $q = x + iy + j\xi$; we want to solve the equation $\gamma q = q$ for the generators of Γ_1. We have the following: If

$$q = \mathbf{a}\, q = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix} q = i(iq+1)^{-1} = \frac{x + i(1-y) + j\xi}{(1-y)^2 + x^2 + \xi^2},$$

then the denominator is 1 and $1 - y = y$, so we obtain the semicircle

$$y = \frac{1}{2}, \quad x^2 + \xi^2 = \frac{3}{4}.$$

If

$$q = \mathbf{b}\, q = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} q = i(iq)^{-1} = \frac{x - iy + j\xi}{y^2 + x^2 + \xi^2},$$

then the denominator is 1 and $-y = y$, so we obtain the semicircle

$$y = 0, \quad x^2 + \xi^2 = 1.$$

If

$$q = \mathbf{c}\, q = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} q = (q+1)(-q)^{-1} = \frac{-x - (x^2 + y^2 + \xi^2) + iy + j\xi}{x^2 + y^2 + \xi^2},$$

then the denominator is 1 and $-x-1=x$, so we obtain the semicircle

$$x=-\frac{1}{2}, \quad y^2+\xi^2=\frac{3}{4}.$$

If

$$q=\mathbf{d}\,q=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} q=-(q)^{-1}=\frac{-x+iy+j\xi}{x^2+y^2+\xi^2},$$

then the denominator is 1 and $-x=x$, so we obtain the semicircle

$$x=0, \quad y^2+\xi^2=1.$$

Now, we can see that the intersections of the semicircles give

$$P=-\frac{1}{2}+\frac{i}{2}+\frac{j}{\sqrt{2}}, \qquad Q=\frac{i}{2}+j\frac{\sqrt{3}}{2}, \qquad R=j, \qquad \text{and} \qquad S=-\frac{1}{2}+j\frac{\sqrt{3}}{2}.$$

Note that all four equations found their solution on the half sphere in $\mathbb{H}^3$ of elements with $x^2+y^2+\xi^2=1$, so we can form a square (sheet) in the sphere with these four points. The space $\underline{E}\Gamma_1$ is then the space generated by the action of the group Γ_1 on that sheet $PQRS$.

As we have mentioned before, the action of any Bianchi group on $\mathbb{H}^3$ can be used to find a presentation for the group and to describe it as an amalgamated product. This was done by Fine [9].

10.3 Bredon cohomology

The Bredon cochain complex with coefficients in the representation ring for the group Γ_1 and the space $X=\underline{E}\Gamma_1$ will be of the form

$$0 \longrightarrow \bigoplus_{\substack{\alpha \\ \text{0-cells}}} R(S_\alpha) \xrightarrow{d^0} \bigoplus_{\substack{\alpha \\ \text{1-cells}}} R(S_\alpha) \xrightarrow{d^1} \bigoplus_{\substack{\alpha \\ \text{2-cells}}} R(S_\alpha) \longrightarrow 0,$$

where the sum runs over representatives of n-cells, the S_α are the corresponding stabilizers, and the differentials are given by restriction of representations. We know that

$$R(A_4)\cong\mathbb{Z}^4, \quad R(S_3)\cong R(S_3')\cong\mathbb{Z}^3, \quad R(D_2)\cong\mathbb{Z}^4, \quad \text{and} \quad R(C_n)\cong\mathbb{Z}^n,$$

so the cochain complex becomes

$$0 \longrightarrow \mathbb{Z}^{4+3+4+3} \xrightarrow{\quad d^0 \quad} \mathbb{Z}^{3+2+2+3} \xrightarrow{\quad d^1 \quad} \mathbb{Z} \longrightarrow 0.$$

Here, d^1 is represented by the matrix $(\,1\;1\;1\;1\;1\;1\;1\;1\;1\;1\,)$, of rank 1, and d^0 by the matrix

$$\left(\begin{array}{cccc:ccc:cccc:ccc}
-1 & 0 & 0 & -1 & 1 & 1 & 0 & & & & & & & \\
0 & -1 & 0 & -1 & 0 & 0 & 1 & & & & & & & \\
0 & 0 & -1 & -1 & 0 & 0 & 1 & & & & & & & \\
\hdashline
 & & & & -1 & 0 & -1 & 1 & 1 & 0 & 0 & & & \\
 & & & & 0 & -1 & -1 & 0 & 0 & 1 & 1 & & & \\
\hdashline
 & & & & & & & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\
 & & & & & & & 0 & -1 & 0 & -1 & 0 & 1 & 1 \\
\hdashline
1 & 0 & 0 & 1 & & & & & & & & -1 & -1 & 0 \\
0 & 0 & 1 & 1 & & & & & & & & 0 & 0 & -1 \\
0 & 1 & 0 & 1 & & & & & & & & 0 & 0 & -1
\end{array}\right),$$

of rank 8. (All the blank spaces mean block of zeros.)

We obtain

$$\mathcal{H}^n_{\Gamma_1}(X;\mathcal{R}) \cong \begin{cases} \mathbb{Z}^6, & n = 0; \\ \mathbb{Z}, & n = 1; \\ 0, & n \geq 2. \end{cases}$$

Note that for Bredon homology, the differentials are in the other direction and are given by induction of representations, so from Frobenius reciprocity we know the matrices associated are the transposed matrices. We will obtain

$$\mathcal{H}^{\Gamma_1}_n(X;\mathcal{R}) \cong \begin{cases} \mathbb{Z}^6, & n = 0; \\ \mathbb{Z}, & n = 1; \\ 0, & n \geq 2. \end{cases}$$

10.4 The congruence subgroup

To define the desired Hecke operator in K-theory we use a particular subgroup associated to a prime in $\mathbb{Z}[i]$, so we take a look at those first.

The units in $\mathbb{Z}[i]$, the Gaussian integers, are 1, -1, i, and $-i$. All the primes, which are the irreducible elements in $\mathbb{Z}[i]$, have either one of the following two forms:

- a or ia, with a prime in $\mathbb{Z}$, such that $|a| \equiv 3 \bmod 4$; or
- $a + ib$, with $a^2 + b^2 = p$ prime in $\mathbb{Z}$.

Regarding the second case, it is known that every prime integer $p \equiv 1 \bmod 4$ can be expressed uniquely as a sum $a^2 + b^2$, hence it corresponds to eight primes in $\mathbb{Z}[i]$ given by $\pm a \pm ib$ and $\pm b \pm ia$. In this way, the second case splits into either $p \equiv 1 \bmod 4$ or the special case $p = 2$, where we have the primes $\pm 1 \pm i$.

The prime integer 2 is the only one that can be written as a square in $\mathbb{Z}[i]$ (so, the only one which ramifies), and this has some consequences in the behaviour of the prime $1 + i$, for instance. One example of this exceptional behaviour will be exposed in the computations for the classifying spaces for proper actions and the isotropy groups.

Let p be any prime in $\mathbb{Z}[i]$. We are interested in the subgroup

$$K = \Gamma_1 \cap g^{-1}\Gamma_1 g, \qquad \text{where } g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

We can describe this subgroup as follows. For any matrix γ we have

$$g^{-1}\gamma g = \begin{pmatrix} 1/p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b/p \\ pc & d \end{pmatrix}.$$

This means that any matrix in K will be of this form. Then

$$K = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \quad : \quad c \in p \cdot \mathbb{Z}[i] \right\},$$

or, as a congruence subgroup,

$$K = \left\{ \gamma \in \Gamma_1 \quad : \quad \gamma \equiv \begin{pmatrix} \cdot & \cdot \\ 0 & \cdot \end{pmatrix} \bmod p \right\}.$$

We can describe the index of K in Γ_1. We need the following lemma.

(The notation $\widetilde{\mathrm{PSL}}_2(\mathbb{F})$ is used for $\mathrm{SL}_2(\mathbb{F})/\{\pm I\}$, since without the tilde it would be the quotient by all the center of the group.)

Lemma 10.1. *For a field with q elements, $\mathbb{F}_q$, the size of the group $\widetilde{\mathrm{PSL}}_2(\mathbb{F}_q)$ is $q(q^2-1)$ if q is even and $q(q^2-1)/2$ if q is odd.*

Proof. The group $\mathrm{SL}_2(\mathbb{F}_q)$ is the kernel of the surjective homomorphism

$$\det : \mathrm{GL}_2(\mathbb{F}_q) \longrightarrow \mathbb{F}_q^*, \quad \text{so} \quad |\mathrm{SL}_2(\mathbb{F}_q)| = |\mathrm{GL}_2(\mathbb{F}_q)|/|\mathbb{F}_q^*| = |\mathrm{GL}_2(\mathbb{F}_q)|/(q-1).$$

The size of $\mathrm{GL}_2(\mathbb{F}_q)$ is equal to the number of bases for $\mathbb{F}_q^2$ over $\mathbb{F}_q$ (there is a non-singular matrix for every pair of linearly independent vectors in $\mathbb{F}_q^2$), which is equal to the number of non-zero vectors in $\mathbb{F}_q^2$ times the number of vectors which are not a multiple of the first one, that is $(q^2-1)(q^2-q)$.

Then, $|\mathrm{SL}_2(\mathbb{F}_q)| = q(q^2-1)$. Finally, since $\widetilde{\mathrm{P}}\mathrm{SL}_2(\mathbb{F}_q) = \mathrm{SL}_2(\mathbb{F}_q)/\{\pm I\}$, we divide by 2 when q is odd and we do not when q is even, because the characteristic of $\mathbb{F}_q$ is 2, so $I = -I$. □

It is known that the quotient $\mathbb{Z}[i]/p$ is a field and is isomorphic to $\mathbb{F}_{|p|}$, where $|p|$ is the norm of p in $\mathbb{Z}[i]$ (the square of its absolute value as a complex number).

Consider the surjective homomorphism

$$\pi : \Gamma_1 = \mathrm{PSL}_2(\mathbb{Z}[i]) \longrightarrow \widetilde{\mathrm{P}}\mathrm{SL}_2(\mathbb{Z}[i]/p).$$

The kernel of π is the group of matrices that are the identity modulo p; the index of this subgroup in Γ_1 is equal to the size of $\widetilde{\mathrm{P}}\mathrm{SL}_2(\mathbb{F}_{|p|})$. And since $\mathrm{Ker}(\pi)$ is contained in the group K, we have

$$(\Gamma_1 : K) = \frac{(\Gamma_1 : \mathrm{Ker}(\pi))}{(K : \mathrm{Ker}(\pi))} = \frac{|\widetilde{\mathrm{P}}\mathrm{SL}_2(\mathbb{F}_{|p|})|}{(K : \mathrm{Ker}(\pi))}.$$

Besides, the index $(K : \mathrm{Ker}(\pi))$ is equal to the size of the quotient group

$$K \,/\, \mathrm{Ker}(\pi) \cong \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \widetilde{\mathrm{P}}\mathrm{SL}_2(\mathbb{F}_{|p|}) \right\},$$

and the size of this group is $|p|(|p|-1)$, if $|p|$ is even, or $|p|(|p|-1)/2$, if $|p|$ is odd. (As before, we do not divide by 2 when $|p|$ is even because $\mathbb{F}_{|p|}$ has characteristic 2.)

Thus, we obtain that

$$(\Gamma_1 : K) = \frac{|p|(|p|^2-1)}{|p|(|p|-1)} = |p| + 1.$$

Furthermore, we can give (left and right) coset representatives for Γ_1 modulo K. There are $|p|$ cosets represented by the matrices

$$\gamma_z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \quad \text{with } z \text{ as representatives of} \quad \mathbb{Z}[i]/p \cong \mathbb{F}_{|p|},$$

and the last is given by the matrix $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now, we wish to compute the Bredon cohomology associated to K. First, note that since K is a subgroup of Γ_1, we can think of $X = \underline{E}\Gamma_1$ as a model for $\underline{E}K$. Then, we need K-orbit representatives for n-cells in X. We can start from the right coset partition

$$\Gamma_1 = \bigsqcup_{K\gamma \in K\backslash\Gamma_1} K\boldsymbol{\gamma}.$$

Note that for any cell $e \subset X$, the Γ_1-orbit of e splits into the union of some K-orbits,

$$\Gamma_1 \cdot e = \bigcup_{K\gamma \in K\backslash\Gamma_1} K\boldsymbol{\gamma} \cdot e,$$

and, after omitting repetitions, the union would be disjoint (apart from the boundaries). To count these repetitions, it is sufficient to find if there exists any $k \in K$ such that

$$\boldsymbol{\gamma}^{-1} k\, \boldsymbol{\gamma}' \in \mathrm{Stab}_{\Gamma_1}(e) \quad \text{for two distinct representatives } \boldsymbol{\gamma},\, \boldsymbol{\gamma}' \text{ of } K\backslash\Gamma_1,$$

in which case we would know that the K-orbits of $\boldsymbol{\gamma} e$ and $\boldsymbol{\gamma}' e$ are the same.

10.4.1 Bredon cohomology

We will focus on the case $p = 1 + i$. With this prime we will compute Bredon cohomology, K-theory and the Hecke operator.

We have that $(\Gamma_1 : K) = |1 + i| + 1 = 3$ and

$$\Gamma_1 = K\boldsymbol{\gamma}_0 \sqcup K\boldsymbol{\gamma}_1 \sqcup K\sigma = K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sqcup K\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sqcup K\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

First we search for repeated K-orbits. These are all we need:

$$\begin{aligned}
\boldsymbol{\gamma}_0^{-1}\begin{pmatrix} -i & i \\ i-1 & 1 \end{pmatrix}\gamma_1 &= \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix} = \mathbf{a}, &\qquad \sigma^{-1}\begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}\gamma_0 &= \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix} = \mathbf{a}, \\
\boldsymbol{\gamma}_0^{-1}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\gamma_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \mathbf{c}^2, &\qquad \sigma^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\gamma_1 &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{c}, \\
\sigma^{-1}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\gamma_0 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \mathbf{b}, &\qquad \sigma^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\gamma_0 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{d},
\end{aligned}$$

$$\gamma_0^{-1}\begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}\gamma_1 = \begin{pmatrix} 1+i & 1 \\ -i & -i \end{pmatrix} = \mathbf{a}^2\mathbf{c}, \quad \sigma^{-1}\begin{pmatrix} i & i \\ 1+i & 1 \end{pmatrix}\gamma_0 = \begin{pmatrix} 1+i & 1 \\ -i & -i \end{pmatrix} = \mathbf{a}^2\mathbf{c},$$

$$\gamma_0^{-1}\begin{pmatrix} i & 0 \\ 1+i & -i \end{pmatrix}\gamma_1 = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix} = \mathbf{ad}, \qquad \sigma^{-1}\begin{pmatrix} -i & i \\ i-1 & 1 \end{pmatrix}\gamma_1 = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix} = \mathbf{a}^2\mathbf{d},$$

$$\gamma_0^{-1}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\gamma_1 = \begin{pmatrix} -i & 0 \\ i & i \end{pmatrix} = \mathbf{bc}, \qquad \sigma^{-1}\begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}\gamma_1 = \begin{pmatrix} i & i \\ 0 & -i \end{pmatrix} = \mathbf{bc}^2.$$

For 0-cells we have

$$\Gamma_1 \cdot P = K \cdot P, \quad \Gamma_1 \cdot Q = K \cdot Q, \quad \Gamma_1 \cdot S = K \cdot S, \quad \text{and}$$

$$\Gamma_1 \cdot R = K \cdot R \ \sqcup \ K\gamma_1 \cdot R = K \cdot R \ \sqcup \ K \cdot \widetilde{R},$$

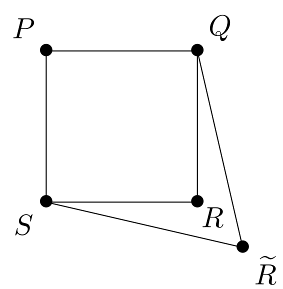
with $\widetilde{R} = \gamma_1 R$. For 1-cells, we have

$$\Gamma_1 \cdot PQ = K \cdot PQ, \quad \Gamma_1 \cdot QR = K \cdot QR \ \sqcup \ K \cdot Q\widetilde{R},$$

$$\Gamma_1 \cdot RS = K \cdot RS \ \sqcup \ K \cdot \widetilde{R}S, \quad \text{and} \quad \Gamma_1 \cdot SP = K \cdot SP,$$

with $Q\widetilde{R} = \gamma_1 QR$ and $\widetilde{R}S = \gamma_1 RS$. And last, the orbit of the 2-cell is not repeated, so there are three 2-cells in the quotient X/K. Let E be the 2-cell $PQRS$.

The quotient space X/K would look like this:



With two 2-cells E and σE with the same boundary, and one other 2-cell $\gamma_1 E$.

The stabilizer of each orbit representative is the intersection between the stabilizer in Γ_1 and the subgroup K, so

$$\mathrm{Stab}_K(P) = A_4 \cap K = \langle \mathbf{ac}, \mathbf{ca}, \mathbf{ac}^2\mathbf{a} \rangle \cong D_2,$$

$$\mathrm{Stab}_K(Q) = S_3 \cap K = \langle \mathbf{a}^2\mathbf{d} \rangle \cong C_2, \quad \mathrm{Stab}_K(R) = D_2 \cap K = \langle \mathbf{bd} \rangle \cong C_2,$$

$$\mathrm{Stab}_K(\widetilde{R}) = \boldsymbol{\gamma}_1 \,\mathrm{Stab}_K(R)\, \boldsymbol{\gamma}_1^{-1} \cong C_2, \quad \mathrm{Stab}_K(S) = S_3' \cap K = \langle \mathbf{bc}^2 \rangle \cong C_2.$$

All the other stabilizers are trivial. The cochain complex becomes

$$0 \longrightarrow \mathbb{Z}^{4+2+2+2+2} \xrightarrow{\ d^0\ } \mathbb{Z}^6 \xrightarrow{\ d^1\ } \mathbb{Z}^3 \longrightarrow 0 \longrightarrow \cdots .$$

Here, d^0 is represented by the matrix

$$\left(\begin{array}{cccc|cc|cc|cc|cc}
-1 & -1 & -1 & -1 & 1 & 1 & & & & & & \\
 & & & & -1 & -1 & 1 & 1 & & & & \\
 & & & & -1 & -1 & & & 1 & 1 & & \\
 & & & & & & -1 & -1 & & & 1 & 1 \\
 & & & & & & & & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & & & & & & & -1 & -1
\end{array}\right),$$

of rank 4, and d^1 by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

of rank 2.

We obtain

$$\mathcal{H}^n_K(X;\mathcal{R}) \cong \begin{cases} \mathbb{Z}^8, & n = 0; \\ 0, & n = 1,\ n > 2; \\ \mathbb{Z}, & n = 2. \end{cases}$$

As before, the homology is computed with the transposed matrices. We have

$$\mathcal{H}_n^K(X;\mathcal{R}) \cong \begin{cases} \mathbb{Z}^8, & n = 0; \\ 0, & n = 1,\ n > 2; \\ \mathbb{Z}, & n = 2. \end{cases}$$

10.4.2 The conjugate

To compute the Hecke operator, we will use the group

$${}_gK := gKg^{-1} = \Gamma_1 \cap g\Gamma_1 g^{-1}.$$

In this case,

$$ {}_gK = \left\{ \gamma \in \Gamma_1 \quad : \quad \gamma \equiv \begin{pmatrix} \cdot & 0 \\ \cdot & \cdot \end{pmatrix} \bmod p \right\}. $$

For the coset representatives, we can take η_z as the transpose of $\boldsymbol{\gamma}_z$ and it works in the same way.

With $p = 1 + i$, we do the same computations as for K, we obtain that $X/{}_gK$ looks the same as X/K, and

$$ \mathrm{Stab}_{{}_gK}(P) = A_4 \cap {}_gK = \langle \mathbf{ac}, \mathbf{ca}, \mathbf{ac}^2\mathbf{a} \rangle \cong D_2, $$

$$ \mathrm{Stab}_{{}_gK}(Q) = S_3 \cap {}_gK = \langle \mathbf{ad} \rangle \cong C_2, \quad \mathrm{Stab}_{{}_gK}(R) = D_2 \cap {}_gK = \langle \mathbf{bd} \rangle \cong C_2, $$

$$ \mathrm{Stab}_{{}_gK}(\widetilde{R}) = \eta_1 \, \mathrm{Stab}_{{}_gK}(R) \, \eta_1^{-1} \cong C_2, \quad \mathrm{Stab}_{{}_gK}(S) = S_3' \cap {}_gK = \langle \mathbf{bc} \rangle \cong C_2. $$

Here, $\eta_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\widetilde{R} = \eta_1 R$. All the other stabilizers are trivial. The cochain complex would be the same and

$$ \mathcal{H}^n_{{}_gK}(X; \mathcal{R}) \cong \begin{cases} \mathbb{Z}^8, & n = 0; \\ 0, & n = 1, \ n > 2; \\ \mathbb{Z}, & n = 2. \end{cases} $$

10.4.3 For other primes

For any other prime in $\mathbb{Z}[i]$, we can use the same process described before to obtain the structure of the quotient space X/K and then the isotropy groups of each orbit.

With `p` a prime in $\mathbb{Z}[i]$, we can compute the orbits of cells with the following GAP program.

```
R := GaussianIntegers;
i := Sqrt(-1);

q := RealPart(p)^2 +ImaginaryPart(p)^2;
max := Maximum( RealPart(p), ImaginaryPart(p) );

## Representatives for Z[i] mod p

list := [];
for x in [ 0 .. max ] do
```

```
    for y in [ 0 .. max ] do
        z := QuotientMod( R, x+y*i, 1, p );
        Add( list, z );
    od;
od;
reprsZi := ShallowCopy( DuplicateFreeList( Concatenation( list
   , -1*list, i*list, -i*list ) ) );

## Coset representatives for Gamma1 mod K

reprs := [ [[1,0],[0,1]], [[0,-1],[1,0]] ];
for j in [ 2 .. Length(reprsZi) ] do
    new := [[1,0],[reprsZi[j],1]];
    Add( reprs, new );
od;

## Generators for Gamma1

a := [[0,i],[i,1]];
b := [[0,i],[i,0]];
c := [[1,1],[-1,0]];
d := [[0,-1],[1,0]];

## Finite subgroups of Gamma1

Ca := Group( [a] ); Cb := Group( [b] ); Cc := Group( [c] ); Cd
    := Group( [d] );
A := Group( [a,c] ); S1 := Group( [a,d] ); S2 := Group( [b,c]
   ); D := Group( [b,d] );

## Lists that will be orbit partition for cells

Porb := []; Qorb := []; Rorb := []; Sorb := [];
PQorb := []; QRorb := []; RSorb := []; SPorb := [];
PQRSorb := [];

## Association of cells with their isotropy groups

cells := [ [Porb, A, "0-cell P"], [Qorb, S1, "0-cell Q"], [
   Rorb, D, "0-cell R"], [Sorb, S2, "0-cell S"],
[PQorb, Ca, "1-cell PQ"], [QRorb, Cd, "1-cell QR"], [RSorb, Cb
```

```
   , "1-cell RS"], [SPorb, Cc, "1-cell SP"],
[PQRSorb, [[[1,0],[0,1]]], "2-cell"] ];

## Making of the partition of orbit representatives

for e in [ 1 .. Length(cells) ] do
    orbits := ShallowCopy(reprs);
    partition := cells[e,1];
    group := cells[e,2];
    count := 0;
    while Length(orbits) > 1 do
        Add( partition, [] );
        count := count+1;
        g1 := orbits[1];
        Add( partition[count], g1 );
        for j in [ 2 .. Length(orbits) ] do
            g2 := orbits[j];
            switch := 0; k := 1;
            while switch = 0 and k <= Length( Elements(group)
               ) do
                x := Elements(group)[k];
                y := g1 * x * g2^(-1);
                if  y[2,1]/p in R then
                    Add( partition[count], g2 );
                    orbits[j] := g1;
                    switch := 1;
                fi;
                k := k+1;
            od;
        od;
        orbits := DuplicateFreeList( orbits );
        Remove( orbits, 1 );
    od;
    if Length(orbits) = 1 then
        Add( partition, [ orbits[1] ] );
    fi;
    Print( "Orbits of ", cells[e,3], ": ", Size( partition ),
       "\n" );
od;
```

The part `Representatives for Z[i] mod p` does not work for `p := 1+i`. In that case we just need `reprsZi := [ 0, 1 ]`.

As said earlier, each orbit in X/Γ_1 splits in at most $(\Gamma_1 : K)$ different orbits in X/K. This GAP program searches which of those $(\Gamma_1 : K)$ orbits are actually the same by the action of K and make that partition in the lists `Porb`, $\ldots$, `PQorb`, $\ldots$.

For example, for `p := 1+i` the output is

```
Orbits of 0-cell P: 1
Orbits of 0-cell Q: 1
Orbits of 0-cell R: 2
Orbits of 0-cell S: 1
Orbits of 1-cell PQ: 1
Orbits of 1-cell QR: 2
Orbits of 1-cell RS: 2
Orbits of 1-cell SP: 1
Orbits of 2-cell: 3
```

And, having made the change `reprsZi := [ 0, 1 ]`, we have

```
gap> Rorb[1];
[ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, -1 ], [ 1, 0 ] ] ]
gap> Rorb[2];
[ [ [ 1, 0 ], [ 1, 1 ] ] ]
```

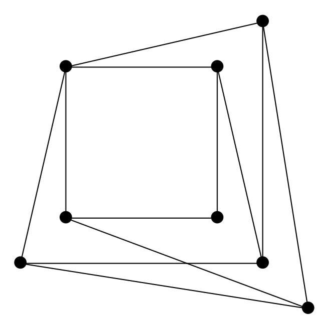
For `p := 2+i` (a prime in $\mathbb{Z}[i]$ with the next smallest norm), we have $(\Gamma_1 : K) = |2+i|+1 = 6$ and

```
Orbits of 0-cell P: 1
Orbits of 0-cell Q: 2
Orbits of 0-cell R: 3
Orbits of 0-cell S: 2
Orbits of 1-cell PQ: 2
Orbits of 1-cell QR: 4
Orbits of 1-cell RS: 4
Orbits of 1-cell SP: 2
Orbits of 2-cell: 6
```

so there are four 1-cells (from QR) joining two 0-cells (from Q) with three 0-cells (from R). To know where the lines should be, we use the partition for those orbits:

```
gap> Qorb;
[  [ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, -1 ], [ 1, 0 ] ], [ [ 1, 0
    ], [ E(4), 1 ] ] ],
    [ [ [ 1, 0 ], [ 1, 1 ] ], [ [ 1, 0 ], [ -1, 1 ] ], [ [ 1,
       0 ], [ -E(4), 1 ] ] ]  ]
gap> Rorb;
[  [ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, -1 ], [ 1, 0 ] ] ],
    [ [ [ 1, 0 ], [ E(4), 1 ] ], [ [ 1, 0 ], [ -E(4), 1 ] ] ],
    [ [ [ 1, 0 ], [ 1, 1 ] ], [ [ 1, 0 ], [ -1, 1 ] ] ]  ]
gap> QRorb;
[  [ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, -1 ], [ 1, 0 ] ] ],
    [ [ [ 1, 0 ], [ E(4), 1 ] ] ],
    [ [ [ 1, 0 ], [ 1, 1 ] ], [ [ 1, 0 ], [ -1, 1 ] ] ],
    [ [ [ 1, 0 ], [ -E(4), 1 ] ] ]  ]
```

Then the quotient space X/K is



where there are two 2-cells in the main square $PQRS$ and the other four are all the other possible combinations.

Now, with any prime in $\mathbb{Z}[i]$ different from $1+i$, the isotropy group of any 0-cell is a cyclic group of order two and the rest are all trivial. This is simply because those are the intersections of the isotropy groups with K. As a consequence, computing the Bredon cohomology is easy. We just need to know the relation between the orbits of cells to compute the differentials.

As part of possible future work, we can develop an algorithm to obtain explicitly the structure of the quotient space X/K, and furthermore the Bredon cohomology groups $\mathcal{H}^n_K(X;\mathcal{R})$.

10.5 K-theory and Hecke operator

From the Atiyah-Hirzebruch spectral sequence we know that, since Bredon cohomology $\mathcal{H}^n_{\Gamma_1}(X;\mathcal{R})$ is trivial for $n \geq 2$,

$$K^n_{\Gamma_1}(X) \cong \begin{cases} \mathbb{Z}^6, & n \text{ even;} \\ \mathbb{Z}, & n \text{ odd.} \end{cases}$$

Similarly, for the subgroup K (and ${}_gK$), the spectral sequence converges already in E_2 so there is a short exact sequence

$$0 \longrightarrow \mathcal{H}^2_K(X;\mathcal{R}) \longrightarrow K^0_K(X) \longrightarrow \mathcal{H}^0_K(X;\mathcal{R}) \longrightarrow 0,$$

but in this case Bredon cohomology groups are free $\mathbb{Z}$-modules, so the extension problem is trivial. Therefore we have

$$K^n_K(X) \quad \cong \quad \begin{cases} \mathbb{Z}^9, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases} \quad \cong \quad K^n_{{}_gK}(X)$$

Note that the K-theory of this groups are free $\mathbb{Z}$-modules, so after dropping the $\mathbb{C}$ we have the isomorphism

$$K^n_G(X) \cong \bigoplus_{[g] \text{ in } G} K^n(X^g)^{C(g)}$$

for Γ_1, K, and ${}_gK$. We compute the Hecke operator there directly, using the definition in Section 6.5.

We need the conjugacy classes of Γ_1 and K (and ${}_gK$). First, note that we only care for conjugacy classes of elements of finite order. Since Γ_1 is an amalgamated product, all elements of finite order are conjugate to those in the factor groups A_4, S_3, S_3', and D_2. So we just need to know which of them are conjugate.

In each group the conjugacy classes are the following:

$$A_4 = \{1\} \sqcup \{\mathbf{a}, \mathbf{c}^2, \mathbf{a}^2\mathbf{c}, \mathbf{c}\mathbf{a}^2\} \sqcup \{\mathbf{c}, \mathbf{a}^2, \mathbf{c}^2\mathbf{a}, \mathbf{a}\mathbf{c}^2\} \sqcup \{\mathbf{ac}, \mathbf{a}\mathbf{c}^2\mathbf{a}, \mathbf{a}^2\mathbf{c}^2\},$$

$$S_3 = \{1\} \sqcup \{\mathbf{a}, \mathbf{a}^2\} \sqcup \{\mathbf{d}, \mathbf{ad}, \mathbf{a}^2\mathbf{d}\}, \quad S_3 = \{1\} \sqcup \{\mathbf{c}, \mathbf{c}^2\} \sqcup \{\mathbf{b}, \mathbf{bc}, \mathbf{bc}^2\},$$

$$D_2 = \{1\} \sqcup \{\mathbf{b}\} \sqcup \{\mathbf{d}\} \sqcup \{\mathbf{bd}\}.$$

Combining them, we obtain six conjugacy classes of finite elements. These are represented by

$$1, \quad \mathbf{a}, \quad \mathbf{ac}, \quad \mathbf{b}, \quad \mathbf{d}, \quad \text{and} \quad \mathbf{bd}.$$

We know there must be six because $\mathcal{H}_0^{\Gamma_1}(X;\mathcal{R}) = \mathbb{Z}^6$, as seen in Theorem 4.1.

For the conjugacy classes in the subgroup K (and ${}_gK$), first we already know the intersections of the factor groups of Γ_1 with K (and ${}_gK$). Also, we know what is the orbit space X/K (and $X/{}_gK$). We can affirm that the elements of finite order in K (and ${}_gK$) are conjugate to those in the stabilizers associated to each orbit of cells in X/K (and $X/{}_gK$). Thus, we have the eight conjugacy classes with representatives

$$1, \quad \mathbf{ac}, \quad \mathbf{ac}^2\mathbf{a}, \quad \mathbf{a}^2\mathbf{c}^2, \quad \mathbf{bc}^2, \quad \mathbf{a}^2\mathbf{d}, \quad \mathbf{bd}, \quad \text{and} \quad \boldsymbol{\gamma}_1\mathbf{bd}\boldsymbol{\gamma}_1^{-1}$$

(and

$$1, \quad \mathbf{ac}, \quad \mathbf{ac}^2\mathbf{a}, \quad \mathbf{a}^2\mathbf{c}^2, \quad \mathbf{bc}, \quad \mathbf{ad}, \quad \mathbf{bd}, \quad \text{and} \quad \eta_1\mathbf{bd}\eta_1^{-1}).$$

We know there must be eight because $\mathcal{H}_0^{K}(X;\mathcal{R}) = \mathbb{Z}^8$ (and $\mathcal{H}_0^{{}_gK}(X;\mathcal{R}) = \mathbb{Z}^8$).

Having Theorem 5.2, we want to identify each copy of $\mathbb{Z}$ in the 0-th K-theory of Γ_1 and K (and ${}_gK$) with one of the summands $K^*(X^g)^{C(g)}$ to split each map. For Γ_1, each conjugacy class corresponds to one copy of $\mathbb{Z}$. In K (and ${}_gK$), the factor corresponding to the trivial conjugacy class is

$$K^0(X^1)^{C_K(1)} \cong K^0(X/K) \cong \mathbb{Z} \oplus \mathbb{Z},$$

since X/K is homotopy equivalent to $\mathbb{S}^2$ (and the same for ${}_gK$).

First, for

$$\mathrm{res} : K^0_{\Gamma_1}(X) \cong \mathbb{Z}^6 \longrightarrow K^0_K(X) \cong \mathbb{Z}^9$$

in conjugacy classes, we have

$$\begin{aligned}
[1] \quad &\longrightarrow \quad [1] \\
[\mathbf{a}] \quad &\longrightarrow \\
[\mathbf{ac}] \quad &\longrightarrow \quad [\mathbf{ac}], \quad [\mathbf{ac}^2\mathbf{a}], \quad [\mathbf{a}^2\mathbf{c}^2] \\
[\mathbf{b}] \quad &\longrightarrow \quad [\mathbf{bc}^2] \\
[\mathbf{d}] \quad &\longrightarrow \quad [\mathbf{a}^2\mathbf{d}] \\
[\mathbf{bd}] \quad &\longrightarrow \quad [\mathbf{bd}], \quad [\boldsymbol{\gamma}_1\mathbf{bd}\boldsymbol{\gamma}_1^{-1}].
\end{aligned}$$

The part corresponding to the identity is a homomorphism $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, where the second component in $K^0(X/K)$ comes from a non-trivial vector bundle, and here the image of a trivial bundle cannot be a non-trivial bundle, so this part would be just an inclusion into the first component.

With the others, the map can be represented by the matrix

$$\begin{array}{cc}
\begin{matrix} \mathbf{1} & \mathbf{a} & \mathbf{ac} & \mathbf{b} & \mathbf{d} & \mathbf{bd} \end{matrix} & \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} &
\begin{matrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{ac} \\ \mathbf{ac^2a} \\ \mathbf{a^2c^2} \\ \mathbf{bc^2} \\ \mathbf{a^2d} \\ \mathbf{bd} \\ \boldsymbol{\gamma_1}\mathbf{bd}\boldsymbol{\gamma_1^{-1}} \end{matrix}
\end{array}.$$

Next, for the map

$$\mathrm{Ad}_g : K^0_K(X) \cong \mathbb{Z}^9 \longrightarrow K^0_{{}^gK}(X) \cong \mathbb{Z}^9,$$

after some computations we have the identifications, given by conjugation $g\,\_\,g^{-1}$,

$$\begin{aligned}
[1] &\longrightarrow [1] \\
[\mathbf{ac}] &\longrightarrow [\mathbf{bc}] \\
[\mathbf{ac^2a}] &\longrightarrow [\mathbf{ad}] \\
[\mathbf{a^2c^2}] &\longrightarrow [\eta_1\mathbf{bd}\eta_1^{-1}] \\
[\mathbf{bc^2}] &\longrightarrow [\mathbf{ac^2a}] \\
[\mathbf{a^2d}] &\longrightarrow [\mathbf{a^2c^2}] \\
[\mathbf{bd}] &\longrightarrow [\mathbf{bd}] \\
[\boldsymbol{\gamma_1}\mathbf{bd}\boldsymbol{\gamma_1^{-1}}] &\longrightarrow [\mathbf{ac}]
\end{aligned}$$

then the matrix is

$$\left(\begin{array}{ccccccccc:c}
\mathbf{1} & \mathbf{1} & \mathbf{ac} & \mathbf{ac}^2\mathbf{a} & \mathbf{a}^c\mathbf{c}^2 & \mathbf{bc}^2 & \mathbf{a}^2\mathbf{d} & \mathbf{bd} & \gamma_1\mathbf{bd}\gamma_1^{-1} & \\
\hdashline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{ac} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{ac}^2\mathbf{a} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \mathbf{a}^2\mathbf{c}^2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{bc} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{ad} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{bd} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \eta_1\mathbf{bd}\eta_1^{-1}
\end{array}\right).$$

At last, for the corestriction map

$$\mathrm{cores} : K^0_{{}_gK}(X) \cong \mathbb{Z}^9 \longrightarrow K^0_{\Gamma_1}(X) \cong \mathbb{Z}^6,$$

we have in conjugacy classes

$$\begin{array}{rcl}
[1] & \longrightarrow & [1] \\
 & \longrightarrow & [\mathbf{a}] \\
[\mathbf{ac}], \quad [\mathbf{ac}^2\mathbf{a}], \quad [\mathbf{a}^2\mathbf{c}^2] & \longrightarrow & [\mathbf{ac}] \\
[\mathbf{bc}] & \longrightarrow & [\mathbf{b}] \\
[\mathbf{ad}] & \longrightarrow & [\mathbf{d}] \\
[\mathbf{bd}], \quad [\eta_1\mathbf{bd}\eta_1^{-1}] & \longrightarrow & [\mathbf{bd}].
\end{array}$$

We want to describe how each centralizer (of the representatives of conjugacy classes in ${}_gK$) in Γ_1 intersects with ${}_gK$. For $\boldsymbol{\gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we want to know when each element commute with $\boldsymbol{\gamma}$, that is, when $\boldsymbol{\gamma}^{-1}k\boldsymbol{\gamma} = k$. Recall that all the calculations must be made modulo $\pm I$, the identity matrix.

First, $(C_{\Gamma_1}(1) : C_{{}_gK}(1)) = (\Gamma_1 : {}_gK) = 3$, and there is a homomorphism

$$K^0(X)^{{}_gK} \cong \mathbb{Z} \oplus \mathbb{Z} \longrightarrow K^0(X)^{\Gamma_1} \cong \mathbb{Z}\,.$$

This part of the transfer map would be the block $(\,3 \;\; 3\,)$.

Now, we have the following:

$$\gamma^{-1}\mathbf{ac}\gamma = \gamma^{-1}\begin{pmatrix} -i & 0 \\ i-1 & i \end{pmatrix}\gamma = \begin{pmatrix} \cdot & b((1-i)b-2id) \\ \cdot & \cdot \end{pmatrix},$$

so $b = 0$ or $(1-i)b = 2id = i(1+i)(1-i)d$ and $b = i(1+i)d$, thus b is always a multiple of $1+i$. This means that $C_{\Gamma_1}(\mathbf{ac}) = C_{{}_gK}(\mathbf{ac})$.

$$\gamma^{-1}\mathbf{ac}^2\mathbf{a}\gamma = \gamma^{-1}\begin{pmatrix} -1 & i-1 \\ i+1 & 1 \end{pmatrix}\gamma = \begin{pmatrix} \cdot & (i-1)(ib+d)(b+d) \\ \cdot & \cdot \end{pmatrix},$$

so $(ib+d)(b+d) = \pm 1$. When it is equal to 1 it means that either $ib+d = b+d = \pm 1$, that implies $b = 0$, or $ib+d = -b-d = \pm i$, that implies $b = \pm(1-i)$. The other case forces b to be a multiple of $1+i$ as well. We have $C_{\Gamma_1}(\mathbf{ac}^2\mathbf{a}) = C_{{}_gK}(\mathbf{ac}^2\mathbf{a})$ as before.

$$\gamma^{-1}\mathbf{a}^2\mathbf{c}^2\gamma = \gamma^{-1}\begin{pmatrix} i & 1+i \\ 0 & -i \end{pmatrix}\gamma = \begin{pmatrix} \cdot & (1+i)((1+i)b+d)d \\ \cdot & \cdot \end{pmatrix},$$

so $((1+i)b+d)d = \pm 1$. When it is equal to 1 it means that either $(1+i)b+d = d = \pm 1$, that implies $b = 0$, or $(1+i)b+d = -d = \pm i$, that implies $b = \pm(-1-i)$. The other case forces b to be a multiple of $1+i$ as well. We have again $C_{\Gamma_1}(\mathbf{a}^2\mathbf{c}^2) = C_{{}_gK}(\mathbf{a}^2\mathbf{c}^2)$.

$$\gamma^{-1}\mathbf{bc}\gamma = \gamma^{-1}\begin{pmatrix} -i & 0 \\ i & i \end{pmatrix}\gamma = \begin{pmatrix} \cdot & -ib(b+2d) \\ \cdot & \cdot \end{pmatrix},$$

so either $b = 0$ or $b = -2d = (1+i)(i-1)d$. This means that $C_{\Gamma_1}(\mathbf{bc}) = C_{{}_gK}(\mathbf{bc})$.

$$\gamma^{-1}\mathbf{ad}\gamma = \gamma^{-1}\begin{pmatrix} -i & 0 \\ -1 & i \end{pmatrix}\gamma = \begin{pmatrix} \cdot & b(b-2id) \\ \cdot & \cdot \end{pmatrix},$$

so either $b = 0$ or $b = 2id = (1+i)^2 d$. This means that $C_{\Gamma_1}(\mathbf{ad}) = C_{{}_gK}(\mathbf{ad})$.

$$\gamma^{-1}\mathbf{bd}\gamma = \gamma^{-1}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\gamma = \begin{pmatrix} -i(ad+bc) & -2ibd \\ 2iac & i(ad+bc) \end{pmatrix},$$

so either $b = 0$ or $d = 0$. If $b = 0$, γ is the identity matrix. In the later case, we have $bc = \pm 1$, but since $ad - bc = 1$, bc must be -1, which leaves only one other possibility not in K, because $a = 0$. Then, $(C_{\Gamma_1}(\mathbf{bd}) : C_{{}_gK}(\mathbf{bd})) = 2$.

$$\gamma^{-1}(\eta_1 \mathbf{bd} \eta_1^{-1})\gamma = \gamma^{-1}\begin{pmatrix} -i & 2i \\ 0 & i \end{pmatrix}\gamma = \begin{pmatrix} \cdot & 2id(d-b) \\ \cdot & \cdot \end{pmatrix},$$

so $d(d - b) = \pm 1$; all cases lead to $b = 0$ or $b = \pm 2i$. Thus, $C_{\Gamma_1}(\eta_1 \mathbf{bd} \eta_1^{-1}) = C_{{}_gK}(\eta_1 \mathbf{bd} \eta_1^{-1})$.

Then, gathering the previous information, the map would be represented by the matrix

$$\begin{array}{c} \begin{array}{ccccccccc} \mathbf{1} & \mathbf{1} & \mathbf{ac} & \mathbf{ac}^2\mathbf{a} & \mathbf{a}^2\mathbf{c}^2 & \mathbf{bc} & \mathbf{ad} & \mathbf{bd} & \eta_1\mathbf{bd}\eta_1^{-1} \end{array} \\ \begin{pmatrix} 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{a} \\ \mathbf{ac} \\ \mathbf{b} \\ \mathbf{d} \\ \mathbf{ac} \end{array}.$$

Finally, we obtain the Hecke operator

$$T_g : K^0_{\Gamma_1}(\underline{E}\Gamma_1) \longrightarrow K^0_{\Gamma_1}(\underline{E}\Gamma_1)$$

given by the matrix

$$\begin{array}{c} \begin{array}{cccccc} 1 & \mathbf{a} & \mathbf{ac} & \mathbf{b} & \mathbf{d} & \mathbf{bd} \end{array} \\ \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ \mathbf{a} \\ \mathbf{ac} \\ \mathbf{b} \\ \mathbf{d} \\ \mathbf{bd} \end{array}.$$

References

[1] M. Artin. *Algebra*. Pearson Prentice Hall, 2011. ISBN: 9780132413770.

[2] M. F. Atiyah. *K-theory*. Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967, pp. v+166+xlix.

[3] Michael Atiyah and Graeme Segal. "On equivariant Euler characteristics". In: *J. Geom. Phys.* 6.4 (1989), pp. 671–677. ISSN: 0393-0440. DOI: 10.1016/0393-0440(89)90032-6.

[4] Paul Baum, Alain Connes and Nigel Higson. "Classifying space for proper actions and K-theory of group C^*-algebras". In: *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*. Vol. 167. Contemp. Math. Amer. Math. Soc., Providence, RI, 1994, pp. 240–291.

[5] P. M. Cohn. "A presentation of SL_2 for Euclidean imaginary quadratic number fields". In: *Mathematika* 15 (1968), pp. 156–163. ISSN: 0025-5793.

[6] P. M. Cohn. "On the structure of the GL_2 of a ring". In: *Inst. Hautes Études Sci. Publ. Math.* 30 (1966), pp. 5–53. ISSN: 0073-8301.

[7] Germán Combariza, Juan Rodríguez and Mario Velásquez. "Induced character in equivariant K-theory and wreath products". In: *arXiv e-prints* (Feb. 2019), arXiv:1902.07097. arXiv: 1902.07097 [math.KT].

[8] Fred Diamond and Jerry Shurman. *A first course in modular forms*. Vol. 228. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+436. ISBN: 0-387-23229-X.

[9] Benjamin Fine. *Algebraic theory of the Bianchi groups*. Vol. 129. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1989. ISBN: 0-8247-8192-9.

[10] Dieter Flöge. "Zur Struktur der PSL_2 über einigen imaginär-quadratischen Zahlringen". In: *Math. Z.* 183.2 (1983), pp. 255–279. ISSN: 0025-5874. DOI: 10.1007/BF01214824.

[11] *GAP – Groups, Algorithms, and Programming, Version 4.10.2.* The GAP Group. 2019. URL: https://www.gap-system.org.

[12] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Sixth. Oxford University Press, Oxford, 2008. ISBN: 978-0-19-921986-5.

[13] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0.

[14] Michio Kuga, Walter Parry and Chih Han Sah. “Group cohomology and Hecke operators”. In: *Manifolds and Lie groups (Notre Dame, Ind., 1980)*. Vol. 14. Progr. Math. Birkhäuser, Boston, Mass., 1981, pp. 223–266.

[15] Min Ho Lee. “Hecke operators on cohomology”. In: *Rev. Un. Mat. Argentina* 50.1 (2009), pp. 99–144. ISSN: 0041-6932.

[16] Wolfgang Lück. “Survey on classifying spaces for families of subgroups”. In: *Infinite groups: geometric, combinatorial and dynamical aspects*. Vol. 248. Progr. Math. Birkhäuser, Basel, 2005, pp. 269–322.

[17] Wolfgang Lück. *Transformation groups and algebraic K-theory*. Vol. 1408. Lecture Notes in Mathematics. Mathematica Gottingensis. Springer-Verlag, Berlin, 1989, pp. xii+443. ISBN: 3-540-51846-0.

[18] Wolfgang Lück and Bob Oliver. “Chern characters for the equivariant K-theory of proper G-CW-complexes”. In: *Cohomological methods in homotopy theory (Bellaterra, 1998)*. Vol. 196. Progr. Math. Birkhäuser, Basel, 2001, pp. 217–247.

[19] Wolfgang Lück and Bob Oliver. “The completion theorem in K-theory for proper actions of a discrete group”. In: *Topology* 40.3 (2001), pp. 585–616. ISSN: 0040-9383. DOI: `10.1016/S0040-9383(99)00077-4`.

[20] John McCleary. *A user's guide to spectral sequences*. Second. Vol. 58. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2001. ISBN: 0-521-56759-9.

[21] Bram Mesland and Mehmet Haluk Sengun. “Hecke operators in KK-theory and the K-homology of Bianchi groups”. In: *arXiv e-prints*, arXiv:1610.06808 (Oct. 2016), arXiv:1610.06808. arXiv: `1610.06808 [math.KT]`.

[22] Guido Mislin and Alain Valette. *Proper group actions and the Baum-Connes conjecture*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003, pp. viii+131. ISBN: 3-7643-0408-1. DOI: `10.1007/978-3-0348-8089-3`.

[23] David E. Muñoz. “Amalgam Decomposition and Cohomology of the Group $SL_2(\mathbb{Z})$ and the Bianchi Groups”. Bachelor's Thesis. Pontificia Universidad Javeriana, 2018.

[24] Alexander Rahm. “(Co)homologies and K-theory of Bianchi groups using computational geometric models”. Theses. Université Joseph-Fourier - Grenoble I, Oct. 2010.

[25] Ruben Jose Sanchez-Garcia. *Equivariant K-homology of the classifying space for proper actions*. Thesis (Ph.D.)–University of Southampton (United Kingdom). ProQuest LLC, Ann Arbor, MI, 2005, (no paging).

[26] Graeme Segal. "Classifying spaces and spectral sequences". In: *Inst. Hautes Études Sci. Publ. Math.* 34 (1968), pp. 105–112. ISSN: 0073-8301.

[27] Mehmet Haluk Şengün. "Arithmetic aspects of Bianchi groups". In: *Computations with modular forms*. Vol. 6. Contrib. Math. Comput. Sci. Springer, Cham, 2014, pp. 279–315. DOI: 10.1007/978-3-319-03847-6_11.

[28] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*. Vol. 11. Publications of the Mathematical Society of Japan. Reprint of the 1971 original, Kanô Memorial Lectures, 1. Princeton University Press, Princeton, NJ, 1994, pp. xiv+271. ISBN: 0-691-08092-5.

[29] Mentor Stafa. *The Mayer-Vietoris spectral sequence*. Department of Mathematics, Tulane University. New Orleans, LA, 2015.