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Classification of rank 1 and 2 affine homogeneous distributions on 3-manifolds

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About this work

This document is the result of a year-long work at the end of a mathematics undergraduate program. It deals with the classification of rank 1 and 2 affine homogeneous distributions on 3-manifolds under a point-affine equivalence. Most of the results come from [1] and [2] as well as the overall idea.

The classification is done using Cartan's reduction method, which basically consists on the reduction of 'geometric objects' into normal forms under some notion of equivalence. In the words of Robert B. Gardner [3] *The goal of the method of equivalence is to find necessary and sufficient conditions in order that 'geometric objects' be 'equivalent'. The word 'equivalent' usually ends up meaning that the geometric objects are mapped onto each other by a class of diffeomorphisms characterized as the set of solutions of a system of differential equations.*

In this case these geometric objects are affine distributions and the notion of equivalence is point-affine equivalence. During the realization of this work [3] was the main reference regarding this technique, as well as [4].

This work is organized as follows: first, the basic notions, definitions and tools are addressed in Chapter 1. Next, the main results are presented in Chapter 2 in the form of four theorems. Finally two examples are given in Chapter 3. Appendix A shows the alternative method followed in the original articles for computing the normal forms of the first theorem. Basic knowledge of manifolds is assumed, in particular vector bundles, vector and covector fields, and also tensor products as well as some background on group theory, specially Lie groups. Most of the definitions were taken from [5] and [6].

The intended audience of this work are mathematic students with some interest in geometry of manifolds that wish to acquire a feeling of Cartan's reduction method or have detailed working examples of the method. Also everyone interested in the actual classification theorems. The objective of this work is twofold:

1. To summarize the results obtained in [1] about the classification of rank 1 and 2 affine homogeneous distributions on 3-manifolds. Also, to provide the background necessary to understand the proofs and overall context of the work, yielding a mostly self-contained document.

2. To proof of the classification theorems exhibiting the maximum amount of detail of the reduction method, particularly following the steps presented in [3].

The main theoretical contribution is Theorem 2.2.2 which is an extension of the work done in [1] or [2]. There are two motivations for the realization of this work: a purely theoretical one and a more broader one regarding the connection between geometry of manifolds and nonlinear control theory.

From a theoretical point of view, the point affine distributions can be directly associated with input-affine control system [1]s. The notion of point-affine equivalence preserves the optimal solutions given by the optimal control problem associated with the input-affine control system. As a result, the classification theorems enable the solution of optimal control problems subject to equivalent dynamical systems solving only the normal form case. This result can be further enhanced by designing control laws for the normal case that can later be applied to any equivalent dynamical system using the appropriate coordinate transformation.

On the other hand, a global motivation is the following: many of the tools and ideas available in geometry of manifolds can be used to some extension in nonlinear control theory. Both theories deal largely with the same problems but from different point of views and with different objectives. Nonetheless, this connection implies that results of geometry of manifolds can be applied to nonlinear control theory problems after some adjustments [7]. Conversely, nonlinear control theory is a very rich and exiting field in which the geometry of manifolds can find many interesting applications [8].

This connection has not been exploited to the maximum because both branches have developed independently, consequently the notions and way of thinking about them is different. This implies that in order to apply results of geometry to nonlinear control theory a good level of understanding of both fields is required, and that is not very common. On the other hand the different points of view and objectives about the same topics yield results that are not compatible without some previous work. The classification theorems of this work are a good example. At first glance they are useless for a control engineer, not only because it is theoretically demanding to understand the hypothesis but also because they do not contribute directly to the synthesis of a controller. However, after some work it is possible to relate concepts such as *bracket-generating distribution* with *controllable dynamical system* from geometry to control theory and apply the theorem to real physical systems.

Furthermore, if some properties useful from a control theory point of view such as stability are proven to be invariant under the defined equivalence suddenly the classification theorem implies that, for a wide range of dynamical system, a single controller in normal form can be applied and like that, the previously abstract and seemingly not very useful theorem transforms into a very powerful tool for the design of control laws.

This is only a very simple and limited example of what I belief to be a much larger bridge between both fields.

Chapter 1

Preliminaries

1.1 Notation and Basic Notions

The following definitions and examples will be used through the text (for details see [5], [6]):

Definition 1.1.1. A **fiber bundle** (E, π, M, S) consists of manifolds E, M and S, and a smooth surjective submersion $\pi : E \to M$ satisfying that each $x \in M$ has an open neighborhood U such that $E|_U := \pi^{-1}(U)$ is diffeomorphic to $U \times S$.

The manifold E is called the *total space*, M the *base space* and S the *standard fiber*. The map π is a called *projection*. If ϕ is the diffeomorphism between $E|_U$ and $U \times S$, then the pair (U, ϕ) is known as a *fiber chart*.

Definition 1.1.2. Let (E, π, M, S) be a fiber bundle and (U_α) an open cover of M, then a collection of compatible fiber charts (U_α, ϕ_α) is called a **fiber bundle atlas**. Given a fiber bundle atlas it is possible to consider of two fiber charts (U_α, ϕ_α) and (U_β, ϕ_β) and the map

$$\begin{aligned} \phi_\alpha \circ \phi_\beta^{-1} : U_\beta \cap U_\alpha \times S &\longrightarrow U_\beta \cap U_\alpha \times S \\ (x, s) &\longmapsto (x, \phi_{\beta\alpha}(x, s)), \end{aligned}$$

where the function $\phi_{\beta\alpha} : U_\beta \cap U_\alpha \times S \to S$ is a smooth function and $\phi_{\beta\alpha}(x, \cdot) : S \to S$ is a diffeomorphism of S for each $x \in U_{\beta\alpha} := U_\beta \cap U_\alpha$. The mappings $\phi_{\beta\alpha}$ are known as the **transition functions** of the bundle (or bundle transition functions).

This transition functions satisfy a cocycle condition, namely:

- $\phi_{\alpha\beta}(x) \circ \phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x)$ for $x \in U_{\alpha\beta\gamma}$,
- $\phi_{\alpha\alpha}(x) = x$ for $x \in U_\alpha$.

In the special case where S is a vector space we obtain the following definition:

Definition 1.1.3. A **vector bundle** is a fiber bundle where the fiber S is an n-dimensional vector space V. In this case the reference to the vector space is omitted and the vector bundle is denoted simply as the triple (E, π, M).

Definition 1.1.4. If the triple (E, π, M) is a vector bundle, then the set $E_x := \pi^{-1}(\{x\})$ is a **fiber** over $x \in M$ and is an n-dimensional vector spaces. The smooth function $\sigma : M \to E$ is called a **section** of the bundle if $\pi \circ \sigma(x) = x$, for all $x \in M$.

There are two very important examples:

Example 1.1.1. The **tangent bundle**: Let

$$E = TM = \big\{(x, v) \mid x \in M,\ v \in T_x M\big\},$$

element $(x, v) \in TM$ is identified with v for convenience. The fiber at x is the space $E_x := \pi^{-1}(\{x\}) = T_x M$, and the sections:

$$\begin{aligned} X : M &\longrightarrow TM \\ x &\longmapsto X_x \in T_x M, \end{aligned}$$

are called vector fields.

Example 1.1.2. The **cotangent bundle**: Let

$$E = T^*M = \big\{(x, \eta) \mid x \in M,\ \eta \in T_x^* M\big\},$$

element $(x, \eta) \in T^*M$ is identified with η for convenience. The fiber at x is the space $E_x := \pi^{-1}(\{x\}) = T_x^* M$ and the sections:

$$\begin{aligned} \Omega : M &\longrightarrow T^*M \\ x &\longmapsto \eta_x \in T_x^* M, \end{aligned}$$

are called covector fields or 1-forms.

Definition 1.1.5. Let (E, π, M) be a vector bundle, then a **subbundle** of E is a vector bundle (D, π_D, M) in which D is a topological subspace of E and π_D is the restriction of π to D such that for each $x \in M$, the subset $D_x = D \cap E_x$ is a linear subspace of E_x and the vector structure of D_x is the one inherited from E_x.

Definition 1.1.6. Given a vector bundle (E, π, M), a **frame** at a point $x \in M$ is an ordered basis for the vector space $E_x = \pi^{-1}(\{x\})$. Define,

$$F_x(E) := \text{set of all frames at } x \in M.$$

Example 1.1.3. For (TM, M, π) we have:

$$F_x := F_x(TM) = \big\{(v_i)_{i=1}^n \mid v_i \in TM,\ \operatorname{span}\{v_i(x)\} = T_x M\big\}.$$

Example 1.1.4. For (T^*M, M, π) we have:

$$F_x^* := F_x(T^*M) = \big\{(v^i)_{i=1}^n \mid v^i \in T^*M,\ \operatorname{span}\{v^i(x)\} = T_x^* M\big\}.$$

The space $F_x(E)$ has a natural left action by the general linear group GL_n:

$$\begin{aligned} GL_n \times F_x &\longrightarrow F_x \\ (g, p) &\longmapsto g \cdot p. \end{aligned}$$

Definition 1.1.7. The **frame bundle** is the triple $(F(E), M, \pi)$ where

$$F(E) := \bigsqcup_{x \in M} F_x(E) = \big\{(x, p) \mid x \in M, p \in F_x(E)\big\},$$

with projection

$$\begin{aligned} \pi : F(E) &\longrightarrow M \\ (x, p) &\longmapsto x. \end{aligned}$$

Again there are two important examples:

Example 1.1.5. The **tangent frame bundle** (or **frame bundle**): $(F(TM), M, \pi)$ where

$$F(TM) := \bigsqcup_{x \in M} F_x(TM) = \big\{(x, p) \mid x \in M,\ p \in F_x(TM)\big\}.$$

Example 1.1.6. The **cotangent frame bundle** (or **coframe bundle**): $(F(T^*M), M, \pi)$ where:

$$F(T^*M) := \bigsqcup_{x \in M} F_x(T^*M) = \big\{(x, p) \mid x \in M,\ p \in F_x(T^*M)\big\}.$$

Definition 1.1.8. Let G be a Lie group and (E, π, M, S) be a fiber bundle. A **G-bundle structure** on the fiber bundle consists of:

1. a left action $\rho : G \times S \to S$ of the Lie group G on the standard fiber S,

2. a fiber bundle atlas (U_α, ϕ_α) whose transition functions $\phi_{\alpha\beta}$ act on S via the G-action. That is, there is a family of smooth mappings $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$ which satisfy that:

   (a) $\varphi_{\alpha\beta}(x)\varphi_{\beta\gamma}(x) = \varphi_{\alpha\gamma}(x)$ for $x \in U_{\alpha\beta\gamma}$,

   (b) $\varphi_{\alpha\alpha}(x) = e$ the identity of G,

   (c) $\phi_{\alpha\beta}(x, s) = \rho(s, \varphi_{\alpha\beta}(x)) = s\varphi_{\alpha\beta}$.

A fiber bundle with a G-bundle structure is called a **G-bundle**.

Definition 1.1.9. A **principal fiber bundle** (P, π, M, G) is a G-bundle where the fiber is a Lie group G and the left action of G on G is just the left translation.

Example 1.1.7. The frame bundle and coframe bundle are principal fiber bundles with Lie group GL_n.

1.2 Method of Equivalence

First denote the coframe bundle by $B(M) := F(T^*M)$. Let $\xi \in B(M)$, i.e. $\xi = (x, (\xi^i)_{i=1}^n)$ is a coframe of M, neglecting the reference to the point $x \in M$ we have $\xi = \{\xi^i\}_{i=1}^n$, where $\xi^i \in T_x^*M$ and $x = \pi(\xi)$.

Definition 1.2.1. The **tautological forms** are 1-forms on $B(M)$ denoted θ^a, $1 \leq a \leq n$ defined by:

$$\begin{array}{rcl} \theta^a : B(M) & \longrightarrow & T_\xi^*(B(M)) \\ \xi & \longmapsto & \theta_\xi^a, \end{array}$$

where

$$\begin{array}{rcl} \theta_\xi^a : T_\xi(B(M)) & \longrightarrow & \mathbb{R} \\ X_\xi & \longmapsto & \xi^a(\mathrm{d}_\xi \pi(X_\xi)). \end{array}$$

Let U be a neighborhood of $x \in M$. Take the coframe field $\eta = \{\eta^i\}_{i=1}^n$ ($\eta^i \in \Omega^1(U)$). Define the function:

$$\begin{array}{rcl} \alpha : \pi^{-1}(U) & \longrightarrow & U \times GL_n \\ (x, \xi) \equiv \xi & \longmapsto & (x, g), \end{array}$$

where $\xi^a = \tilde{g}_b^a \eta_x^b$, $[\tilde{g}_b^a] = g^{-1}$. Now consider the pullbacks of these 1-forms, using the map

$$\begin{array}{rcl} \mathrm{d}\pi_{(x,\xi)}^* : T_x^*M & \longrightarrow & T_{(x,\xi)}^*(\pi^{-1}(U)) \\ \eta_x^a & \longmapsto & \mathrm{d}\pi_{(x,\xi)}^*(\eta_x^a) = \bar{\eta}_{x,\xi}^a. \end{array}$$

We have the definition:

$$\theta_{(x,\xi)}^a = \tilde{g}_b^a \bar{\eta}_{(x,\xi)}^b = \tilde{g}_b^a \mathrm{d}\pi_{(x,\xi)}^*(\eta_x^b).$$

Or in vector form:

$$\theta_{(x,\xi)} = g^{-1} \mathrm{d}\pi_{(x,\xi)}^*(\eta_x). \tag{1.1}$$

Definition 1.2.2. A G**-structure** $P \to M$ is a principal subbundle of $\pi : B(M) \to M$ with the group structure $G \subset GL_n$, that is: $(P, \pi|_P, M, G)$ is itself a principal bundle where

$$P = \bigsqcup_{x \in M} U_x,$$

and the U_x are vector subspaces of the fibers $B(M)_x = F_x(T^*M)$.

Given a coframe of tautological forms $\theta = (\theta^a)_{a=1}^n$ we define the **structure equations** as:

$$\mathrm{d}\theta^a = \omega_b^a \wedge \theta^b + T_{bc}^a \theta^b \wedge \theta^c.$$

where ω_b^a are the Maurer-Cartan forms. The functions $T_{bc}^a : P \to \mathbb{R}$ are called the **torsion functions**, and the map

$$T : P \to \Lambda^2 \mathbb{R}^n \otimes \mathbb{R}^n \ , \ \xi \mapsto T_{bc}^a(\xi) e^b \wedge e^c \otimes e_a,$$

where $\{e_i\}$ is the canonical base form $\mathbb{R}$ is called the **torsion**.

Definition 1.2.3. The **Spencer operator** δ in an operator defined on $T_1^2(\mathbb{R}^n)$ by:

$$\begin{aligned} \delta : T_1^2(\mathbb{R}^n) &\longrightarrow \Lambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n \\ t^a_{bc} &\longmapsto t^a_{[bc]} = \frac{1}{2}(t^a_{bc} - t^a_{cb}). \end{aligned}$$

Using the Spencer operator, we define the space:

$$\mathcal{T} = \frac{\Lambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n}{\delta(\mathfrak{g} \otimes (\mathbb{R}^n)^*)},$$

where $\mathfrak{g}$ is the Lie algebra of G. Using this space we define the **structure function**:

$$\begin{aligned} \mathcal{C} : P &\longrightarrow \mathcal{T} \\ \xi &\longmapsto [T^a_{bc}(\xi)]. \end{aligned}$$

The structure equations are said to be homogeneous if all the structure functions T^a_{bc} are equal to constants. On the other hand the group G acts on $\mathcal{T}$ by:

$$\begin{aligned} \rho(g) : \mathcal{T} &\longrightarrow \mathcal{T} \\ T^a_{bc} &\longmapsto \tilde{g}^a_r T^r_{pq} g^p_b g^q_c. \end{aligned}$$

The orbits of this action relate with the torsion functions in the following form: Consider two sets of structure equations

$$\mathrm{d}\theta^a = \omega^a_b \wedge \theta^b + T^a_{bc}\theta^b \wedge \theta^c, \qquad \mathrm{d}\tilde{\theta}^a = \tilde{\omega}^a_b \wedge \tilde{\theta}^b + \tilde{T}^a_{bc}\tilde{\theta}^b \wedge \tilde{\theta}^c,$$

corresponding to the same $G-$structure. Then the torsion functions are related by the orbits of the G action by

$$\rho(g)(T^a_{bc} e_a \otimes e^b \wedge e^c) = \tilde{g}^a_r T^r_{pq} g^p_b g^q_c e_a \otimes e^b \wedge e^c \quad \Longleftrightarrow \quad \tilde{T}^a_{bc} = \tilde{g}^a_r T^r_{pq} g^p_b g^q_c. \tag{1.2}$$

Given a $G-$structure $P \to M$, let $\mathcal{T} = \bigsqcup \mathcal{T}_\alpha$ be the partition of $\mathcal{T}$ into orbits by the action of G. Assume that the structure function $\mathcal{C}$ takes values only on one orbit $\mathcal{T}_0$. Then fix $\tau_0 \in \mathcal{T}_0$ and define:

$$\hat{P} = \{\xi \in P \mid \mathcal{C}(\xi) = \tau_0\}, \tag{1.3}$$

$$\hat{G} = \{g \in G \mid \rho(g)\tau_0 = \tau_0\}. \tag{1.4}$$

Here, $\hat{P}$ is the total space of a principal $\hat{G}$-subundle of P (which is itself a subbundle of $B(M)$ with the group structure GL_n). This procedure of obtaining $\hat{G}$-structure $\hat{P} \to M$ from the $G-$structure $P \to M$ is known as the **Cartan Reduction**.

1.3 Distributions

Definition 1.3.1. Let M be an n-dimensional manifold, then a **distribution** of rank k on M, is a rank-k subbundle D of TM, $D_x \subset T_xM$ is a linear subspace of dimension k for each $x \in M$ and the distribution D can be thought of as

$$D = \bigcup_{x \in M} D_x.$$

Being a subbundle, each point x of M has a neighborhood U on which there are smooth vector fields $X_1, X_2, \dots, X_k : U \to TM$ such that $X_1|_x, X_2|_x, \dots X_k|_x$ are a basis for D_x for every $x \in U$.

Definition 1.3.2. Let D be a smooth distribution. A nonempty immersed submanifold $N \subset M$ is called an **integral manifold** of D if $T_xN = D_x$ at each point $x \in N$. A smooth distribution D on M is said to be **integrable** if each point of M is contained in an integral manifold of D. Furthermore, the distribution D is **completely integrable** if for each $x \in M$ there exists a neighborhood U such that the first k coordinate vector fields $\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^k$ span D_U. A completely integrable distribution is also known as a **Frobenius distribution**.

1.4 Point-Affine Equivalence

Definition 1.4.1. A **dynamical system** is a triple $(\mathcal{X}, \mathcal{U}, \Gamma)$ where $\mathcal{X}$ is an open subset of $\mathbb{R}^n$ called the state space, $\mathcal{U}$ is an open subset of $\mathbb{R}^m$ and Γ is a smooth function that satisfies

$$\begin{aligned} \Gamma : \mathcal{X} \times \mathcal{U} &\longrightarrow \mathbb{R}^n \\ (x, u) &\longmapsto \Gamma(x, u) = \dot{x}. \end{aligned}$$

Such a dynamical system is said to have n states, m inputs and represent the dynamics given by $\dot{x} = \Gamma(x, u)$ where $x \in \mathbb{R}^n$ are the states and $u \in \mathbb{R}^m$ are the controls.

Definition 1.4.2. An **input-affine system** is a dynamical system where the function Γ has the form

$$\Gamma(x, u) = F(x) + G(x)u.$$

Representing the dynamics

$$\dot{x} = F(x) + G(x)u, \tag{1.5}$$

where $F : \mathcal{X} \to \mathcal{X}$ is a smooth vector function known as the **drift vector** and $G : \mathcal{X} \to \mathbb{R}^{n \times m}$ is smooth matrix function.

Definition 1.4.3. A **rank-s affine distribution** $\mathcal{F}$ on an n-dimensional manifold $\mathcal{X}$ is a smoothly-varying family of s-dimensional, affine linear subspaces $\mathcal{F}_x \subset T_x\mathcal{X}$. The distribution $\mathcal{F}$ is **strictly affine** if none of the subspaces $\mathcal{F}_x \subset T_x\mathcal{X}$ are linear subspaces. Each affine distribution $\mathcal{F}$ has a corresponding **direction distribution**:

$$L_{\mathcal{F}} = \{X_1 - X_2 \mid X_1, X_2 \in \mathcal{F}\}$$

Let $\mathcal{F}$ be an affine distribution on a manifold $\mathcal{X}$. Let $\mathcal{F}$ also denote the sheaf of smooth vector fields on $\mathcal{X}$ which are local sections of $\mathcal{F}$. The flag of subsheaves

$$\mathcal{F} = \mathcal{F}^1 \subset \mathcal{F}^2 \subset \cdots \subset T\mathcal{X}$$

May be defined in the recursive way: Let $\mathcal{F}^1 = \mathcal{F}$ and then for $i \geq 1$

$$\mathcal{F}^{i+1} = \mathcal{F}^i + [\mathcal{F}, \mathcal{F}^i]$$

Given a point x in the manifold $\mathcal{X}$, the flag of subsheaves gives a flag of affine subspaces of $T_x\,\mathcal{X}$:

$$\mathcal{F}_x^1 \subset \mathcal{F}_x^2 \subset \cdots \subset T_x\,\mathcal{X}\,.$$

Definition 1.4.4. Denote $\mathcal{F}^\infty = \cup_{i\geq 1}\mathcal{F}^i \subset T\,\mathcal{X}$. Then the smallest integer $r = r(x)$ such that $\mathcal{F}_x^r = \mathcal{F}_x^\infty$ is called the **step** of the distribution at x. Let $n_i(x) = \dim(\mathcal{F}_x^i)$. The **growth vector** of $\mathcal{F}$ at x is the list of integers $(n_1(x), n_2(x), \ldots, n_r(x))$, where r is the step of $\mathcal{F}$ at x. The distribution $\mathcal{F}$ is **bracket-generating** if $\mathcal{F}^\infty = T\,\mathcal{X}$. On the other hand $\mathcal{F}$ is **almost bracket-generating** is $\mathrm{rank}(\mathcal{F}^\infty) = n-1$ and for each $x \in \mathcal{X}$ and any $v \in \mathcal{F}_x$, $\mathrm{span}(v(x), (L_{\mathcal{F}^\infty})_x) = T_x\,\mathcal{X}$. Furthermore the distribution $\mathcal{F}$ has **constant type** if:

- The growth vector of $\mathcal{F}$ is constant on $\mathcal{X}$.
- For any section v of $\mathcal{F}$, $\dim(\mathrm{span}(v(x), (L_{\mathcal{F}^i})_x))$ is constant on $\mathcal{X}$ for all i.

Definition 1.4.5. A **point affine distribution** $\mathcal{F}$ on a manifold $\mathcal{X}$ is an affine distribution $\mathcal{F}$ on $\mathcal{X}$, together with a **distinguished vector field** $v_0 \in \mathcal{F}$.

Any input affine dynamical system

$$\dot{x} = v_0(x) + \sum_{i=1}^{s} v_i(x)u^i,$$

has a canonical identification with the affine distribution $\mathcal{F}$ those fibers are

$$\mathcal{F}_x = \Big\{v_0(x) + \sum_{i=1}^{s} \lambda_i v_i(x) \mid \lambda_1 \in \mathbb{R} \ \text{ for } i = 1, 2, \ldots, s\Big\}.$$

Definition 1.4.6. Given two input affine systems

$$\dot{x} = a_0(x) + \sum_{i=1}^{s} a_i(x)u^i, \qquad\qquad \dot{y} = b_0(y) + \sum_{j=1}^{s} b_j(y)v^j,$$

defined on the manifolds $\mathcal{X}$ and $\mathcal{Y}$, with local coordinate representation of the vector fields:

$$a_l(x) = \sum_{k=1}^{n} a_l^k(x)\frac{\partial}{\partial x^k}, \qquad\qquad b_l(x) = \sum_{k=1}^{n} b_l^k(x)\frac{\partial}{\partial y^k}, \qquad l = 0, 1, \ldots, n$$

They are **locally point-affine equivalent** if there exist a diffeomorphism $\psi : \mathcal{X} \longrightarrow \mathcal{Y}$ that satisfies

$$\psi_*(a_0(x)) = b_0(\psi(x)), \qquad \psi_*(a_i(x)) = \sum_{j=1}^{s} \lambda_i^j(x) b_j(\psi(x)) \quad i = 1, 2, \ldots, s$$

The condition of being locally point-affine equivalent translates in local coordinates to:

$$b_0^k(\psi(x)) = \sum_{q=1}^{n} \frac{\partial \psi^k}{\partial x^q}(x) a_0^q(x), \qquad \sum_{j=1}^{s} \lambda_i^j(x) b_j^k(\psi(x)) = \sum_{q=1}^{n} \frac{\partial \psi^k}{\partial x^q}(x) a_i^q(x), \quad \begin{array}{l} k = 1, 2, \ldots, n \\ i = 1, 2, \ldots, s. \end{array}$$

Chapter 2

Classification Theorems

2.1 Rank-1 Distributions in 3-Manifolds

Theorem 2.1.1 (Local classification of rank-1 strictly affine distributions on 3-manifolds)**.** Let $\mathcal{F}$ be a rank 1 strictly affine point-affine distribution of constant type on a manifold M of dimension 3. Then:

1. If $\mathcal{F}$ is almost bracket-generating, then in a sufficiently small neighborhood of any point $x \in M$, there exist local coordinates (x^1, x^2, x^3) such that

$$\mathcal{F} = \left(\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^3}\right) + \operatorname{span}\left(\frac{\partial}{\partial x^3}\right)$$

   where J is an arbitrary function on M.

2. If $\mathcal{F}$ is bracket-generating and $L_{\mathcal{F}^2}$ is Frobenius, then in a sufficiently small neighborhood of any point $x \in M$, there exist local coordinates (x^1, x^2, x^3) such that

$$\mathcal{F} = \left(x^2\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^3}\right) + \operatorname{span}\left(\frac{\partial}{\partial x^3}\right)$$

   where J is an arbitrary function on M.

3. If $\mathcal{F}$ is bracket-generating and $L_{\mathcal{F}^2}$ is not Frobenius, then in a sufficiently small neighborhood of any point $x \in M$, there exist local coordinates (x^1, x^2, x^3) such that

$$\mathcal{F} = \left((1 + x^3 J)\frac{\partial}{\partial x^1} + J\frac{\partial}{\partial x^2} + JH\frac{\partial}{\partial x^3}\right) + \operatorname{span}\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right)$$

   where J and H are arbitrary functions on M satisfying $\frac{\partial H}{\partial x^1} \neq 0$.

Proof. Let M be a 3-manifold and $\mathcal{F}$ be a rank 1 point-affine distribution on M. The problem is to classify the possible point affine distributions that are of constant type and strictly affine.

Start with a local framing (v_1, v_2, v_3) on M, the distinguished vector field will be v_1 and v_2 will generate the distributions, that is:

$$\mathcal{F} = v_1 + \mathrm{span}(v_2).$$

In order for another local framing $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ to be equivalent it must generate the same distribution, which implies that

$$\tilde{v}_1 = v_1, \qquad \tilde{v}_2 = b_2 v_2, \qquad \tilde{v}_3 = a_3 v_1 + b_3 v_2 + c_3 v_3,$$

where $b_2 \neq 0$ and $c_3 \neq 0$.

These conditions extend to the coframings (η^1, η^2, η^3) and $(\tilde{\eta}^1, \tilde{\eta}^2, \tilde{\eta}^3)$ in M by:

$$\begin{pmatrix} \tilde{\eta}^1 \\ \tilde{\eta}^2 \\ \tilde{\eta}^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix}^{-1} \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix}. \tag{2.1}$$

The problem of classifying equivalent coframings in M can be solved using the Cartan reduction method in the lifted space $B(M)$ via the map $g^{-1}\mathrm{d}\pi$ as in (1.1) with the corresponding principal bundle B_0 with structure group G_0 defined by:

$$B_0 = \{(x, \xi_x) \in B(M) \mid x \in \mathbb{R}^3,\ \xi_x \in F_x(T^* \mathbb{R}^3),\ \text{i.e., } \xi_x = (\theta^1_x, \theta^2_x, \theta^3_x)^T\},$$
$$G_0 = \left\{ \begin{pmatrix} 1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} \middle| \ b_2 c_3 \neq 0 \right\}.$$

Step 0: G_0-Structure $B_0 \longrightarrow \mathbb{R}^3$

Let us consider the principal subbundle

$$B_0 = \{(x, \xi_x) \in B(M) \mid x \in \mathbb{R}^3,\ \xi_x \in F_x(T^* \mathbb{R}^3),\ \text{i.e. } \xi_x = (\theta^1_x, \theta^2_x, \theta^3_x)^T\}, \tag{2.2}$$

and the corresponding Lie group and corresponding Lie algebra:

$$G_0 = \left\{ \begin{pmatrix} 1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} \middle| \ b_2 c_3 \neq 0 \right\} \qquad \Longleftrightarrow \qquad \mathfrak{g}_o = \left\{ \begin{pmatrix} 0 & 0 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix} \right\}. \tag{2.3}$$

With this Lie algebra the structure equations are

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \\ \mathrm{d}\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T^1_{23} & T^1_{13} & T^1_{12} \\ T^2_{23} & T^2_{13} & T^2_{12} \\ T^3_{23} & T^3_{13} & T^3_{12} \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^1 \wedge \theta^3 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{2.4}$$

In order to reduce the subbundle we need to compute the orbits of the G_0-action:

1. A basis for $\mathfrak{g}_o$ is:
$$\mathfrak{g}_o = \{e_2 \otimes e^2, e_1 \otimes e^3, e_2 \otimes e^3, e_3 \otimes e^3\}.$$

2. Thus a basis for $\mathfrak{g}_o \otimes \mathbb{R}^3$ is:
$$\mathfrak{g}_o \otimes \mathbb{R}^3 = \{e_2 \otimes e^2 \otimes e^i, e_1 \otimes e^2 \otimes e^i, e_2 \otimes e^3 \otimes e^i, e_3 \otimes e^3 \otimes e^i\}_{i=1,2,3}.$$

3. The action of the Spencer operator δ in each element of the basis is given by:
$$\begin{aligned} \delta : \mathfrak{g}_o \otimes \mathbb{R}^3 &\longrightarrow \Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3 \\ \\ e_1 \otimes e^3 \otimes e^1 &\longmapsto e_1 \otimes e^3 \wedge e^1 \\ e_1 \otimes e^3 \otimes e^2 &\longmapsto e_1 \otimes e^3 \wedge e^2 \\ e_1 \otimes e^3 \otimes e^3 &\longmapsto 0 \\ e_2 \otimes e^2 \otimes e^1 &\longmapsto e_2 \otimes e^2 \wedge e^1 \\ e_2 \otimes e^2 \otimes e^2 &\longmapsto 0 \\ e_2 \otimes e^2 \otimes e^3 &\longmapsto e_2 \otimes e^2 \wedge e^3 \\ e_2 \otimes e^2 \otimes e^1 &\longmapsto e_2 \otimes e^2 \wedge e^1 \\ e_2 \otimes e^3 \otimes e^1 &\longmapsto e_2 \otimes e^3 \wedge e^1 \\ e_2 \otimes e^3 \otimes e^2 &\longmapsto e_2 \otimes e^3 \wedge e^2 \\ e_2 \otimes e^3 \otimes e^3 &\longmapsto 0 \\ e_3 \otimes e^3 \otimes e^1 &\longmapsto e_3 \otimes e^3 \wedge e^1 \\ e_3 \otimes e^3 \otimes e^2 &\longmapsto e_3 \otimes e^3 \wedge e^2 \\ e_3 \otimes e^3 \otimes e^3 &\longmapsto 0, \end{aligned}$$
hence
$$\begin{aligned} \delta(\mathfrak{g}_o \otimes \mathbb{R}^3) = \langle & e_1 \otimes e^1 \wedge e^2, e_1 \otimes e^2 \wedge e^3, e_2 \otimes e^1 \wedge e^2, e_2 \otimes e^1 \wedge e^3, \\ & e_2 \otimes e^2 \wedge e^3, e_3 \otimes e^1 \wedge e^3, e_3 \otimes e^2 \wedge e^3\rangle. \end{aligned}$$

4. This implies that
$$\mathcal{T}_0 = \frac{\Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3}{\delta(\mathfrak{g}_o \otimes \mathbb{R}^3)} = \langle e_1 \otimes e^1 \wedge e^2, e_3 \otimes e^1 \wedge e^2\rangle,$$
and the structure function is:
$$\begin{aligned} \mathcal{C} : B_0 &\longrightarrow \mathcal{T}_0 \\ \xi &\longmapsto T^1_{12}(\xi) e_1 \otimes e^1 \wedge e^2 + T^3_{12}(\xi) e_3 \otimes e^1 \wedge e^2. \end{aligned}$$

5. In order to compute the orbits of $\mathcal{T}_0$ calculate the G_0-action in the generators: Let $g \in G_0$, then
$$\begin{aligned} \rho(g)(e_1 \otimes e^1 \wedge e^2) &= b_2 e_1 \otimes e^1 \wedge e^2, \\ \rho(g)(e_3 \otimes e^1 \wedge e^2) &= \frac{-a_3}{c_3} b_2 e_1 \otimes e^1 \wedge e^2 + \frac{b_2}{c_3} e_3 \otimes e^1 \wedge e^2. \end{aligned}$$
According to (1.2) the torsion functions transform as
$$\tilde{T}^1_{12} = b_2 T^1_{12} - \frac{a_3 b_2}{c_3} T^3_{12}, \qquad \tilde{T}^3_{12} = \frac{b_2}{c_3} T^3_{12}.$$

Step 1: G_1-Structure $B_1 \longrightarrow \mathbb{R}^3$

Take $\tau_0 = e_3 \otimes e^1 \wedge e^2$. Using (1.3) results in the G_1-Structure $B_1 \to \mathbb{R}^3$ with principal subbundle:

$$B_1 = \{(x, \xi_x) \in B_0 \mid T^1_{12}(\xi) = 0,\ T^3_{12}(\xi) = 1\}. \tag{2.5}$$

Replacing in (1.4), the corresponding Lie structure group G_1 and its Lie algebra $\mathfrak{g}_1$ are:

$$G_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} \middle| \; c_3 \neq 0 \right\} \qquad \Longleftrightarrow \qquad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_3 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix} \right\}. \tag{2.6}$$

With this Lie algebra and principal subbundle the structure equations are

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \\ \mathrm{d}\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_3 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T^1_{23} & T^1_{13} & 0 \\ T^2_{23} & T^2_{13} & T^2_{12} \\ T^3_{23} & T^3_{13} & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^1 \wedge \theta^3 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{2.7}$$

In order to reduce the subbundle we need to compute the orbits of the G_1-action:

1. A basis for $\mathfrak{g}_1$ is:
$$\mathfrak{g}_1 = \{e_2 \otimes e^2 + e_3 \otimes e^3, e_2 \otimes e^3\}.$$

2. Thus a basis for $\mathfrak{g}_1 \otimes \mathbb{R}^3$ is:
$$\mathfrak{g}_1 \otimes \mathbb{R}^3 = \{e_2 \otimes e^2 \otimes e^i + e_3 \otimes e^3 \otimes e^i, e_2 \otimes e^3 \otimes e^i\}_{i=1,2,3}.$$

3. The action of the Spencer operator δ in each element of the basis is:
$$\delta : \mathfrak{g}_1 \otimes \mathbb{R}^3 \longrightarrow \Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3$$
$$\begin{aligned}
e_2 \otimes e^2 \otimes e^1 + e_3 \otimes e^3 \otimes e^1 &\longmapsto -e_2 \otimes e^1 \wedge e^2 - e_3 \otimes e^1 \wedge e^3 \\
e_2 \otimes e^2 \otimes e^2 + e_3 \otimes e^3 \otimes e^2 &\longmapsto -e_3 \otimes e^2 \wedge e_3 \\
e_2 \otimes e^2 \otimes e^3 + e_3 \otimes e^3 \otimes e^3 &\longmapsto e_2 \otimes e^2 \wedge e^3 \\
e_2 \otimes e^3 \otimes e^1 &\longmapsto -e_2 \otimes e^1 \wedge e^3 \\
e_2 \otimes e^3 \otimes e^2 &\longmapsto -e_2 \otimes e^2 \wedge e^3 \\
e_2 \otimes e^3 \otimes e^3 &\longmapsto 0,
\end{aligned}$$
hence
$$\delta(\mathfrak{g}_1 \otimes \mathbb{R}^3) = \langle e_2 \otimes e^1 \wedge e^2 + e_3 \otimes e^1 \wedge e^3, e_3 \otimes e^2 \wedge e^3, e_2 \otimes e^2 \wedge e^3, e_2 \otimes e^1 \wedge e^3 \rangle.$$

4. This implies that
$$\mathcal{T}_1 = \frac{\Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3}{\delta(\mathfrak{g}_1 \otimes \mathbb{R}^3)} = \langle e_2 \otimes e^1 \wedge e^2 - e_3 \otimes e^1 \wedge e^3, e_1 \otimes e^1 \wedge e^2, e_1 \otimes e^1 \wedge e^3, e_1 \otimes e^2 \wedge e^3, e_3 \otimes e^1 \wedge e^2 \rangle$$
and the structure function is:
$$\begin{aligned}
\mathcal{C} : B_1 &\longrightarrow \mathcal{T}_1 \\
\xi &\longmapsto T^2_{12}(\xi)(e_2 \otimes e^1 \wedge e^2 - e_3 \otimes e^1 \wedge e^3) + T^1_{12}(\xi) e_1 \otimes e^1 \wedge e^2 + \\
&\qquad T^1_{13}(\xi) e_1 \otimes e^1 \wedge e^3 + T^1_{23}(\xi) e_1 \otimes e^2 \wedge e^3 + T^3_{12}(\xi) e_3 \otimes e^1 \wedge e^2.
\end{aligned}$$

5. In order to compute the orbits of $\mathcal{T}_1$ calculate the G_1-action in the generators, let $g \in G_1$.

$$\begin{aligned}
\rho(g)(e_2 \otimes e^1 \wedge e^2 - e_3 \otimes e^1 \wedge e^3) &= e_2 \otimes e^1 \wedge e^2 - e_3 \otimes e^1 \wedge e^3,\\
\rho(g)(e_1 \otimes e^1 \wedge e^2) &= c_3 e_1 \otimes e^1 \wedge e^2 + b_3 e_1 \otimes e^1 \wedge e^3,\\
\rho(g)(e_1 \otimes e^1 \wedge e^3) &= c_3 e_1 \otimes e^1 \wedge e^3,\\
\rho(g)(e_1 \otimes e^2 \wedge e^3) &= c_3^2 e_1 \otimes e_2 \wedge e^3,\\
\rho(g)(e_3 \otimes e^1 \wedge e^2) &= \frac{-b_3}{c_3}(e_2 \otimes e^1 \wedge e^2 - e_3 \otimes e^1 \wedge e^3) + e_3 \otimes e^1 \wedge e^2.
\end{aligned}$$

According to (1.2) the torsion functions transform as

$$\tilde{T}^1_{13} = c_3 T^1_{13}, \qquad \tilde{T}^1_{23} = c_3^2 T^1_{23}, \qquad \tilde{T}^3_{13} = T^3_{13} + \frac{b_3}{c_3}.$$

Step 2: G_2-Structure $B_2 \longrightarrow \mathbb{R}^3$ Take $\tau_1 = e_3 \otimes e^1 \wedge e^2$. Using (1.3) results in the G_2-Structure $B_2 \to \mathbb{R}^3$ with principal subbundle

$$B_2 = \{(x, \xi_x) \in B_1 \mid T^3_{13}(\xi) = 0\}. \tag{2.8}$$

Replacing in (1.4), the corresponding Lie structure group G_2 and its Lie algebra $\mathfrak{g}_2$ are:

$$G_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \middle| \; c_3 \neq 0 \right\} \qquad \Longleftrightarrow \qquad \mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_3 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} \right\}. \tag{2.9}$$

With this Lie algebra and principal subbundle the structure equations are

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \\ \mathrm{d}\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_3 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T^1_{23} & T^1_{13} & 0 \\ T^2_{23} & T^2_{13} & T^2_{12} \\ T^3_{23} & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^1 \wedge \theta^3 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{2.10}$$

In order to reduce the subbundle we need to compute the orbits of the G_2-action:

1. A basis for $\mathfrak{g}_2$ is:

$$\mathfrak{g}_2 = \{e_2 \otimes e^2 + e_3 \otimes e^3\}.$$

2. Thus a basis for $\mathfrak{g}_2 \otimes \mathbb{R}^3$ is:

$$\mathfrak{g}_2 \otimes \mathbb{R}^3 = \{e_2 \otimes e^2 \otimes e^i + e_3 \otimes e^3 \otimes e^i\}_{i=1,2,3}.$$

3. The action of the Spencer operator δ in each element of the basis is given by:

$$\delta : \mathfrak{g}_2 \otimes \mathbb{R}^3 \longrightarrow \Lambda^2(\mathbb{R}^3) \otimes \mathbb{R}^3$$

$$\begin{aligned}
e_2 \otimes e^2 \otimes e^1 + e_3 \otimes e^3 \otimes e^1 &\longmapsto -e_2 \otimes e^1 \wedge e^2 - e_3 \otimes e^1 \wedge e^3\\
e_2 \otimes e^2 \otimes e^2 + e_3 \otimes e^3 \otimes e^2 &\longmapsto -e_3 \otimes e^2 \wedge e_3\\
e_2 \otimes e^2 \otimes e^3 + e_3 \otimes e^3 \otimes e^3 &\longmapsto e_2 \otimes e^2 \wedge e^3,
\end{aligned}$$

hence:

$$\delta(\mathfrak{g}_2 \otimes \mathbb{R}^3) = \langle e_2 \otimes e^1 \wedge e^2 + e_3 \otimes e^1 \wedge e^3, e_3 \otimes e^2 \wedge e^3, e_2 \otimes e^2 \wedge e^3 \rangle.$$

4. This implies that

$$\mathcal{T}_2 := \frac{\Lambda^2(\mathbb{R}^3)\otimes\mathbb{R}^3}{\delta(\mathfrak{g}_2\otimes\mathbb{R}^3)} = \langle e_2\otimes e^1\wedge e^2 - e_3\otimes e^1\wedge e^3, e_1\otimes e^1\wedge e^2, e_1\otimes e^1\wedge e^3,$$
$$e_1\otimes e^2\wedge e^3, e_2\otimes e^1\wedge e^3, e_3\otimes e^1\wedge e^2\rangle.$$

and the structure function is:

$$\begin{aligned}\mathcal{C} : B_2 &\longrightarrow \mathcal{T}_2\\ \xi &\longmapsto T^2_{12}(\xi)(e_2\otimes e^1\wedge e^2 - e_3\otimes e^1\wedge e^3) + T^1_{12}(\xi)e_1\otimes e^1\wedge e^2 + T^1_{13}(\xi)e_1\otimes e^1\wedge e^3\\ &\quad T^1_{23}(\xi)e_1\otimes e^2\wedge e^3 T^2_{13}(\xi)e_2\otimes e^1\wedge e^3 T^3_{12}(\xi)e_3\otimes e^1\wedge e^2.\end{aligned}$$

5. In order to compute the orbits of $\mathcal{T}_2$ calculate the G_2-action in the generators: Let $g\in G_2$.

$$\begin{aligned}\rho(g)(e_2\otimes e^1\wedge e^2 - e_3\otimes e^1\wedge e^3) &= e_2\otimes e^1\wedge e^2 - e_3\otimes e^1\wedge e^3,\\ \rho(g)(e_1\otimes e^1\wedge e^2) &= e_1\otimes e^1\wedge e^2,\\ \rho(g)(e_1\otimes e^1\wedge e^3) &= c_3 e_1\otimes e^1\wedge e^3,\\ \rho(g)(e_1\otimes e^2\wedge e^3) &= c_3^2 e_1\otimes e^2\wedge e^3,\\ \rho(g)(e_2\otimes e^1\wedge e^3) &= e_2\otimes e^1\wedge e^3,\\ \rho(g)(e_3\otimes e^1\wedge e^2) &= e_3\otimes e^1\wedge e^2.\end{aligned}$$

According to (1.2) the torsion functions transform as

$$\tilde{T}^1_{13} = c_3 T^1_{13}, \qquad \tilde{T}^1_{23} = c_3^2 T^1_{23}, \qquad \tilde{T}^2_{13} = T^2_{13}.$$

Before performing the last reduction we need to consider three cases:

1. $T^1_{13} = T^1_{23} = 0$ in this case it is not necessary to reduce further.

2. $T^1_{23} = 0$ and $T^1_{13} \neq 0$ in this case it is necessary to reduce one more time to get a $G-$structure where $G = \{e\}$ (also noted $e-$structure).

3. $T^1_{23} \neq 0$ in this case one more reduction is required but the resulting $G-$Structure is not trivial.

The idea now is to use the resulting structure equations to find normal forms for the coframes and frames in M. Recalling that the structure equations $\mathrm{d}\theta^a = \omega^a_b\wedge\theta^b + T^a_{bc}\theta^b\wedge\theta^c$ are defined for coframes in the principal subbundles given by each reduction it is necessary to express them in terms of coframes in M using the pullback of the section π.

Since the system of equations is highly underdetermined there are many frames and coframes that satisfy the corresponding structure equations in M (all equivalent), if a simple solution is not clear then a general form is assumed and conditions over the coordinate functions are deduced from the structure equations, finally the simplest functions that satisfy this conditions are chosen.

Case 1: $T^1_{13} = T^1_{23} = 0$

Since the torsion functions T^2_{23}, T^2_{12} and T^3_{23} do not appear in the orbits of the action they are not affected by the reduction so they are free to take any value, thus for simplicity we can make them vanish. Hence the structure equations are

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \\ \mathrm{d}\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_3 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & T^2_{13} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^1 \wedge \theta^3 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{2.11}$$

These forms are defined on the lifted space B_2, thus in order to find the corresponding forms in M consider their pullbacks via the section $\sigma : M \to B_2$:

$$\begin{aligned} \mathrm{d}\eta_1 &= 0, \\ \mathrm{d}\eta_2 &= \bar{\gamma}_3 \wedge \eta^2 + \bar{T}^2_{13}\eta^1 \wedge \eta^3, \\ \mathrm{d}\eta_3 &= \bar{\gamma}_3 + \bar{T}^2_{12}\eta^1 \wedge \eta^2. \end{aligned}$$

Since η^1 is exact we can chose a coordinate x^1 on M such that $\eta^1 = \mathrm{d}x^1$. From the structure equations $\mathrm{d}\eta^3 \equiv 0 \mod \{\eta^3, \mathrm{d}x^1\}$ so it is not exact and must be a combination of η^3 and $\mathrm{d}x^1$. Taking the two remaining coordinates the simplest form that satisfies these conditions is

$$\eta^3 = \mathrm{d}x^2 - x^3\mathrm{d}x^1.$$

Finally the third structure equation implies that

$$\eta^2 = \mathrm{d}x^3 + B\mathrm{d}x^1 + C(\mathrm{d}x^2 - x^3\mathrm{d}x^1).$$

For some functions B and C on M, from the structure equations it follows that

$$\bar{\gamma}_2 = C\mathrm{d}x^1 + D(\mathrm{d}x^2 - x^3\mathrm{d}x^1),$$

where D is another function in M and

$$C = \frac{1}{2}\partial_3 B, \qquad D = \frac{1}{2}\partial^2_{33} B.$$

In summary the coframing is given by the one-forms:

$$\begin{aligned} \eta^1 &= \mathrm{d}x^1, \\ \eta^2 &= \mathrm{d}x^3 + B\mathrm{d}x^1 + \frac{1}{2}\partial_3 B(\mathrm{d}x^2 - x^3\mathrm{d}x^1), \\ \eta^3 &= \mathrm{d}x^2 - x^3\mathrm{d}x^1. \end{aligned}$$

Now, it is easy to calculate the corresponding vector fields of the dual framing:

$$\begin{aligned} v_1 &= \frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - B\frac{\partial}{\partial x^3}, \\ v_2 &= \frac{\partial}{\partial x^3}, \\ v_3 &= \frac{\partial}{\partial x^2} - \frac{1}{2}\partial_3 B\frac{\partial}{\partial x^3}. \end{aligned}$$

Setting $J = -B$, the distribution is given by:

$$\mathcal{F} = v_1 + \mathrm{span}(v_2) = \Big(\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - J\frac{\partial}{\partial x^3}\Big) + \mathrm{span}\Big(\frac{\partial}{\partial x^3}\Big).$$

Finally, from the structure equations

$$T^2_{13} = \partial_2 J - \frac{1}{2}(\partial^2_{13}J + x^3\partial_{32}J + J\partial^2_{33}J) + \frac{1}{4}(\partial_3 J)^2.$$

Case 2: $T^1_{23} = 0$ and $T^1_{13} \neq 0$

In this case since $T^1_{13} \neq 0$ it is possible to continue the reduction, recall that we had the orbit

$$\rho(g)(e_1 \otimes e^1 \wedge e^3) = c_3 e_1 \otimes e^1 \wedge e^3.$$

Thus let $\tau_2 = e_1 \otimes e^1 \wedge e^3$. Using (1.3) results in the G_3-Structure $B_3 \to \mathbb{R}^3$ with principal subbundle

$$B_3 = \{(x, \xi_x) \in B_2 \mid T^1_{13}(\xi) = 1\}. \tag{2.12}$$

Replacing in (1.4), the corresponding Lie structure group G_3 and its Lie algebra $\mathfrak{g}_3$ are:

$$G_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \qquad \Longleftrightarrow \qquad \mathfrak{g}_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{2.13}$$

With this Lie algebra and principal subbundle the structure equations are

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \\ \mathrm{d}\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ T^2_{23} & T^2_{13} & T^2_{12} \\ T^3_{23} & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^1 \wedge \theta^3 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{2.14}$$

As it was done before use the section $\pi : M \to B_3$ to pullback this forms to obtain the structure equations in M:

$$\begin{aligned} \mathrm{d}\eta^1 &= \eta^1 \wedge \eta^3, \\ \mathrm{d}\eta^2 &= \bar{T}^2_{12}\eta^1 \wedge \eta^2 + \bar{T}^2_{13}\eta^1 \wedge \eta^3 + \bar{T}^2_{23}\eta^2 \wedge \eta^3, \\ \mathrm{d}\eta^3 &= \eta^1 \wedge \eta^2 + \bar{T}^2_{12}\eta^1 \wedge \eta^3. \end{aligned} \tag{2.15}$$

Now, note that η^1 is a closed one-form, i.e. $\mathrm{d}\eta^1 \equiv 0 \mod \eta^1$ but is not exact, so in principle we can take η^1 in only one coordinate and make the function not defined in $x^2 = 0$ so that it is not exact. Thus, take the coordinates x^1 and x^2 and define

$$\eta^1 = \frac{1}{x^2}\mathrm{d}x^1.$$

Then the first structure function implies that

$$\mathrm{d}\eta^1 = \frac{1}{(x^2)^2}\mathrm{d}x^1 \wedge \mathrm{d}x^2 = \frac{1}{x^2}\eta^1 \wedge \mathrm{d}x^2 = \eta^1 \wedge \eta^3,$$

hence

$$\eta^3 = \frac{1}{x^2}\mathrm{d}x^2 \mod \mathrm{d}x^1.$$

Also note that $\mathrm{d}\eta^3 \neq 0 \mod \eta^3$, the easiest way to guarantee this is to make the function in the x^1 coordinate depend on x^3 so that $\mathrm{d}\eta^3$ has components in the three coordinates while η^3 not. And the simplest function is the linear one, so we make

$$\eta^3 = \frac{1}{x^2}\Big(\mathrm{d}x^2 - \frac{x^3}{x^2}\mathrm{d}x^1\Big).$$

With η^1 and η^3 we can use the structure equations to arrive at

$$\eta^2 = \frac{1}{x^2}\Big(\mathrm{d}x^3 + \frac{1}{x^2}B\mathrm{d}x^1 + \frac{1}{x^2}C\big(\mathrm{d}x^2 - \frac{x^3}{x^2}\mathrm{d}x^1\big)\Big),$$

for some functions B and C on M.

The structure equations also imply that $C = \frac{1}{2}(x^2\partial_3 B - x^3)$. In summary the coframe is

$$\begin{aligned}
\eta^1 &= \frac{1}{x^2}\mathrm{d}x^1,\\
\eta^2 &= \frac{1}{x^2}\Big(\mathrm{d}x^3 + \frac{1}{x^2}B\mathrm{d}x^1 + \frac{1}{x^2}\frac{1}{2}(x^2\partial_3 B - x^3)\big(\mathrm{d}x^2 - \frac{x^3}{x^2}\mathrm{d}x^1\big)\Big),\\
\eta^3 &= \frac{1}{x^2}\Big(\mathrm{d}x^2 - \frac{x^3}{x^2}\mathrm{d}x^1\Big).
\end{aligned}$$

The dual framing is

$$\begin{aligned}
v_1 &= x^2\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - B\frac{\partial}{\partial x^3},\\
v_2 &= x^2\frac{\partial}{\partial x^3},\\
v_3 &= x^2\frac{\partial}{\partial x^2} - \frac{1}{2}(x^2\partial_3 B - x^3)\frac{\partial}{\partial x^3}.
\end{aligned}$$

Let $J = -B$. Then the distribution is given by

$$\mathcal{F} = v_1 + \operatorname{span}(v_2) = \Big(x^2\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^3}\Big) + \operatorname{span}\Big(\frac{\partial}{\partial x^3}\Big).$$

Finally from the structure equations

$$\begin{aligned}
T^2_{12} =& \frac{1}{2x^2}(x^2\partial_3 J - 3x^3),\\
T^2_{13} =& \frac{1}{4(x^2)^2}\Big(3(x^2)^2 - 6x^2J + 4(x^2)^2\partial_2 J + 2x^2x^3\partial_3 J - 2(x^2)^3\partial^2_{13}J\\
&- 2(x^2)^2x^3\partial^2_{23}J + (x^2)^2(\partial_3 J)^2 - 2(x^2)^2J\partial^2_{33}J\Big),\\
T^2_{23} =& \frac{1}{2}(1 - x^2\partial^2_{33}J).
\end{aligned}$$

Case 3: $T^1_{23} \neq 0$

Since $T^1_{23} \neq 0$ it is possible to continue the reduction, in this case the corresponding orbit of interest is

$$\rho(g)(e_1 \otimes e^2 \wedge e^3) = c_3^2 e_1 \otimes e^2 \wedge e^3.$$

Thus let $\tau_2 = e_1 \otimes e^2 \wedge e^3$. Using (1.3) results in the G_3-Structure $B_3 \to \mathbb{R}^3$ with principal subbundle

$$B_3 = \{(x, \xi_x) \in B_2 \mid T^1_{23}(\xi) = 1\}.$$

Replacing in (1.4), the corresponding Lie structure group G_1 and its Lie algebra $\mathfrak{g}_1$ are:

$$G_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| \det(g) = 1 \right\} \qquad \Longleftrightarrow \qquad \mathfrak{g}_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

With this Lie algebra and principal subbundle the structure equations are

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \\ \mathrm{d}\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} 1 & T^1_{13} & 0 \\ T^2_{23} & T^2_{13} & T^2_{12} \\ T^3_{23} & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^1 \wedge \theta^3 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{2.16}$$

As before use the section $\pi : M \to B_3$ to pullback this forms to obtain the structure equations in M:

$$\begin{aligned}
\mathrm{d}\eta^1 &= \bar{T}^1_{13}\eta^1 \wedge \eta^3 + \eta^2 \wedge \eta^3, \\
\mathrm{d}\eta^2 &= \bar{T}^2_{12}\eta^1 \wedge \eta^2 + \bar{T}^2_{13}\eta^1 \wedge \eta^3 + \bar{T}^2_{23}\eta^2 \wedge \eta^3, \\
\mathrm{d}\eta^3 &= \eta^1 \wedge \eta^2 + \bar{T}^2_{12}\eta^1 \wedge \eta^3 + \bar{T}^3_{23}\eta^2 \wedge \eta^3.
\end{aligned}$$

Since $\mathrm{d}\eta^1 \neq 0 \mod \eta^1$ as in case 2 we make

$$\eta^1 = \mathrm{d}x^1 - x^3 \mathrm{d}x^2.$$

Now set $T^1_{13} = B$, then the first structure equation implies that

$$(B\eta^1 + \eta^2) \wedge \eta^3 = \mathrm{d}x^2 \wedge \mathrm{d}x^3.$$

Consequently,

$$\begin{aligned}
\eta^2 &= (-B(\mathrm{d}x^1 - x^3\mathrm{d}x^2)\lambda^{-1}\mathrm{d}x^2 + C(H\mathrm{d}x^2 - \mathrm{d}x^3)), \\
\eta^3 &= \lambda(H\mathrm{d}x^2 - \mathrm{d}x^3).
\end{aligned}$$

For functions λ, C and H on M $(\lambda \neq 0)$.

The third structure equation implies $-\lambda^2 \partial_1 H = 1$ so $\partial_1 H < 0$ and

$$\lambda = \frac{1}{\sqrt{-\partial_1 H}}.$$

In summary the coframing is

$$\begin{aligned}
\eta^1 &= \mathrm{d}x^1 - x^3\mathrm{d}x^2,\\
\eta^2 &= -\sqrt{-\partial_1 H}\mathrm{d}x^2 - B(\mathrm{d}x^1 - x^3\mathrm{d}x^2) - C(H\mathrm{d}x^2) - \mathrm{d}x^3,\\
\eta^3 &= \frac{1}{\sqrt{-\partial_1 H}}(H\mathrm{d}x^2 - \mathrm{d}x^3).
\end{aligned}$$

The dual framing is

$$\begin{aligned}
v_1 &= \frac{\partial}{\partial x^1} - \frac{B}{\sqrt{-\partial_1 H}}\Big(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\Big),\\
v_2 &= \frac{1}{\sqrt{-\partial_1 H}}\Big(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\Big),\\
v_3 &= -\sqrt{-\partial_1 H}\frac{\partial}{\partial x^3} + C\Big(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\Big).
\end{aligned}$$

Let $J = \frac{-B}{\sqrt{-\partial_1 H}}$, the distribution is

$$\mathcal{F} = \Big(\frac{\partial}{\partial x^1} - \frac{B}{\sqrt{-\partial_1 H}}\Big(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\Big)\Big) + \operatorname{span}\Big(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\Big)$$

Finally replacing in the structure equations we get the following relations:

$$\begin{aligned}
C =& \frac{1}{2\partial_1 H}\sqrt{-\partial_1 H}\Bigg(\Big(\partial_1 H\partial_3 H - H\partial_{13}^2 H - \partial_{12}^2 H\Big)J\\
&- (1 + x^3 J)\partial_{11}^2 H - (\partial_2 J + x^3\partial_1 J + H\partial_3 J)\partial_1 H\Bigg),\\
T_{12}^2 =& \frac{1}{2}(\partial_2 J + x^3\partial_1 J + H\partial_3 J + J\partial_3 H),\\
T_{13}^2 =& C^2\partial_1 H - \partial_1 H\partial_3 J + J^2\partial_1 H + \frac{1}{2\sqrt{-\partial_1 H}}\Bigg(2C\Big(\partial_1 H\partial_2 J + J\partial_{12}^2 H + \partial_{11}^2 H\\
&+ 2x^3\partial_1 H\partial_1 J + x^3 J\partial_{11}^2 H + 2HJ\partial_{13}^2 H - 2J\partial_1 H\partial_3 H\Big)\\
&- 2x^3 J\partial_1 C\partial_1 H - 2\partial_1 C\partial_1 H - 2HJ\partial_3 C\partial_1 H - 2J\partial_2 C\partial_1 H\Bigg),\\
T_{23}^2 =& -\Big(\partial_2 C + x^3\partial_1 C + H\partial_3 C + C\partial_3 H\Big) - \frac{1}{2\sqrt{-\partial_1 H}}\Big(\partial_{13}^2 H + 2J\partial_1 H\Big),\\
T_{23}^3 =& \frac{1}{2\partial_1 H\sqrt{-\partial_1 H}}\Big(x^3\partial_{11}^2 H - 2\partial_1 H\partial_3 H + \partial_{12}^2 H + H\partial_{13}^2 H\Big).
\end{aligned}$$

□

Theorem 2.1.2 (Normal form classification of rank-1 strictly affine distributions on 3-manifolds)**.** Let $\mathcal{F}$ be a rank-1, strictly-affine, bracket-generating or almost bracket-generating point-affine distribution of constant type on a 3-dimensional manifold M. If the structure functions corresponding to the distribution $\mathcal{F}$ are homogeneous, then $\mathcal{F}$ is locally point-affine equivalent to

$$\mathcal{F} = v_1 + \operatorname{span}(v_2), \tag{2.17}$$

where the vectors v_1 and v_2 are one of the following:

- If $\mathcal{F}$ is almost bracket-generating, then there are two options:
  - Case 1:
$$\begin{aligned} v_1(x) &= \frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + (c_2x^2 + c_3x^3)\frac{\partial}{\partial x^3}, \\ v_2(x) &= \frac{\partial}{\partial x^3}. \end{aligned}$$
  - Case 2:
$$\begin{aligned} v_1(x) &= \frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + c_3x^3\frac{\partial}{\partial x^3}, \\ v_2(x) &= \frac{\partial}{\partial x^3}. \end{aligned}$$
- If $\mathcal{F}$ is bracket-generating and $L_{\mathcal{F}^2}$ is Frobenius, then there is one case:
  - Case 3:
$$\begin{aligned} v_1(x) &= x^2\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + x^2\left(\frac{3}{2}\left(\frac{x^3}{x^2}\right) + c_1\right)\frac{\partial}{\partial x^3}, \\ v_2(x) &= x^2\frac{\partial}{\partial x^3}. \end{aligned}$$
- If $\mathcal{F}$ is bracket-generating and $L_{\mathcal{F}^2}$ is not Frobenius, then there are three cases:
  - Case 4:
$$\begin{aligned} v_1(x) &= \frac{\partial}{\partial x^1} + c_1\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \epsilon(x^1 + c_2x^3)\frac{\partial}{\partial x^3}\right), \\ v_2(x) &= \epsilon\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \epsilon(x^1 + c_2x^3)\frac{\partial}{\partial x^3}\right). \end{aligned}$$
  - Case 5:
$$\begin{aligned} v_1(x) &= \frac{\partial}{\partial x^1} + \frac{c_1\cos(c_3x^1)}{\sqrt{\epsilon c_3(c_3(x^3)^2 + c_4)}}\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right), \\ v_2(x) &= \epsilon\Big(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\Big). \end{aligned}$$

    Where $H = \left((c_3(x^3)^2 + c_4)\tan(c_3x^1) + F_{20}(x^2)\sqrt{c_3(x^3)^2 + c_4}\right)$.

– Case 6:

$$v_1(x) = \frac{\partial}{\partial x^1} + \frac{c_1\cos(c_3x^1)}{\sqrt{\epsilon c_3(c_3(x^3)^2 - c_4)}}\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right),$$
$$v_2(x) = \epsilon\Big(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\Big).$$

Where $H = \left((-c_3(x^3)^2 + c_4)\tan(c_3x^1) + F_{20}(x^2)\sqrt{c_3(x^3)^2 - c_4}\right)$.

Proof. Start with the three cases of Theorem 2.1.1:

- **First Case**: the distribution $\mathcal{F}$ is

$$\mathcal{F} = \left(\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^3}\right) + \operatorname{span}\left(\frac{\partial}{\partial x^3}\right),$$

where J is an arbitrary function on M. The corresponding coframe is

$$\eta^1 = \mathrm{d}x^1,$$
$$\eta^2 = (1 + x^3\partial_3 J)\mathrm{d}x^1 - \partial_3 J\mathrm{d}x^2 + \mathrm{d}x^3,$$
$$\eta^3 = -x^3\mathrm{d}x^1 + \mathrm{d}x^2.$$

The corresponding structure equations are not homogeneous, hence a modification needs to be made, take

$$v_2 = G^{-1/2}(x)\frac{\partial}{\partial x^3}.$$

Then the frame becomes

$$v_1 = \frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^3},$$
$$v_2 = G^{-1/2}\frac{\partial}{\partial x^3},$$
$$v_3 = G^{-1/2}\frac{\partial}{\partial x^2} - (v_1(G^{-1/2}) - G^{-1/2}\partial_3 J)\frac{\partial}{\partial x^3}.$$

With corresponding coframe

$$\eta_1 = \mathrm{d}x^1,$$
$$\eta_2 = \Big(x^3G(G^{-1/2}\partial_3 J - v_1(G^{-1/2})) - JG^{-1/2}\Big)\mathrm{d}x^1 + G\Big(v_1(G^{-1/2}) - G^{-1/2}\partial_3 J\Big)\mathrm{d}x^2 + G^{-1/2}\mathrm{d}x^3,$$
$$\eta_3 = -x^3G^{-1/2}\mathrm{d}x^1 + G^{-1/2}\mathrm{d}x^2.$$

Now determine the coordinate transformations that preserve this coframe, let $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ be new coordinates related to the old ones by:

$$x^1 = \phi_1(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \qquad x^2 = \phi_2(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \qquad x^3 = \phi_3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3).$$

The first covector gives

$$\eta^1 = \mathrm{d}x^1 = \tilde{\eta}_1 = \mathrm{d}\tilde{x}^1 \Longrightarrow x^1 = \tilde{x}^1 + a.$$

The third covector

$$\begin{aligned}\eta_3 &= -x^3 G^{\text{-}1/2}\mathrm{d}x^1 + G^{\text{-}1/2}\mathrm{d}x^2 \\ &= -\phi_2 G^{\text{-}1/2}\mathrm{d}\tilde{x}^1 + G^{\text{-}1/2}(\partial_1\phi_2\mathrm{d}\tilde{x}^1 + \partial_2\phi_2\mathrm{d}\tilde{x}^2 + \partial_3\phi_2\mathrm{d}\tilde{x}^3) \\ &= -\tilde{x}^3\tilde{G}^{\text{-}1/2}\mathrm{d}\tilde{x}^1 + \tilde{G}^{\text{-}1/2}\mathrm{d}\tilde{x}^2.\end{aligned}$$

Hence $\partial_3\phi_2 = 0 \Longrightarrow x^2 = \phi_2(\tilde{x}^1, \tilde{x}^2)$, also $\tilde{G}^{\text{-}1/2} = G^{\text{-}1/2}\partial_2\phi_2$. Finally in a similar way making $\eta^2 = \tilde{\eta}^2$ and using the previous results gives

$$\begin{aligned}\phi_3 &= \partial_1\phi_2 + \tilde{x}^3\partial_2\phi_2, \\ \partial_3\tilde{J} = \partial_3 J &= (\partial_2\phi)^{-1}(2\partial^2_{22}\phi_2\tilde{x}^3 + 2\partial^2_{12}\phi_2).\end{aligned} \tag{2.18}$$

Rename $\phi_2 = \phi$. The third structure equation gives

$$\mathrm{d}\eta^3 = \frac{\partial_3 G}{2G^{\frac{3}{2}}}\eta^2 \wedge \eta^3 \qquad \mathrm{mod}\ \eta^1.$$

Hence $\frac{\partial_3 G}{2G^{\frac{3}{2}}}$ must be a constant $-c_1$. There are two cases depending on whether c_1 is zero or nonzero.

1. If $c_1 = 0$ then $\partial_3 G = 0$ thus $G(x^1, x^2, x^3) = G_0(x^1, x^2)$. The allowed local change of coordinates imply that

   $$\partial_2\phi(\tilde{x}^1, \tilde{x}^2) = \frac{1}{G_0(\tilde{x}^1 + a, \phi(\tilde{x}^1, \tilde{x}^2))}.$$

   It is possible to normalize $\tilde{G}_0(\tilde{x}^1, \tilde{x}^2) = 1$. In order for this relation to hold under coordinate transformation $\tilde{G}^{\frac{1}{2}} = G^{\frac{1}{2}}\partial_2\phi$ implies that $\partial_2\phi = 1$ hence

   $$\phi(\tilde{x}^1, \tilde{x}^2) = \tilde{x}^2 + \phi_0(\tilde{x}^1).$$

   If $G(x^1, x^2, x^3) = 1$ then the third structure equations becomes

   $$\mathrm{d}\eta^3 = \eta^1 \wedge \eta^2 + \partial_3 J\eta^1 \wedge \eta^3.$$

   Hence $\partial_3 J = c_3$ and so $J(x^1, x^2, x^3) = x^3 + J_0(x^1, x^2)$. The second structure equation for $\mathrm{d}\eta^2$ gives

   $$\mathrm{d}\eta^2 = \partial_2 J_0\eta^1 \wedge \eta^2.$$

   Hence $\partial_2 J_0 = c_2$ and $J_0(x^1, x^2) = x^2 + J_1(x^1)$. Replacing in (2.18) gives

   $$\tilde{J}_1(\tilde{x}^1) = J_1(\tilde{x}^1 + a) - (\phi_0''(\tilde{x}^1) - c_3\phi_0'(\tilde{x}^1) - c_2\phi_0(\tilde{x}^1)).$$

   The normalization $\tilde{J}_1(\tilde{x}^1) = 0$ is preserved for functions $\phi_0(\tilde{x}^1)$ that satisfy

   $$0 = \phi_0''(\tilde{x}^1) - c_3\phi_0'(\tilde{x}^1) - c_2\phi_0(\tilde{x}^1). \tag{2.19}$$

   In the end the coordinate transformations that preserve the structure equations are

   $$x^1 = \tilde{x}^1 + a, \qquad x^2 = \tilde{x}^2 + \phi_0(\tilde{x}^1), \qquad x^3 = \tilde{x}^3 + \phi_0'(\tilde{x}^1).$$

   Where ϕ_0' satisfies (2.19).

2. If $c_1 \neq 0$ then
$$G(x^1, x^2, x^3) = \frac{1}{(c_1 x^3 + G_0(x^1, x^2))^2}.$$
The allowed change of coordinates must satisfy
$$\partial_1 \phi(\tilde{x}^1, \tilde{x}^2) = \frac{1}{c_1} G_0(\tilde{x}^1 + a, \phi(\tilde{x}^1, \tilde{x}^2)).$$
Normalizing $G_0 = 0$ this condition is preserved if $\phi(\tilde{x}^1, \tilde{x}^2) = \phi_0(\tilde{x}^1)$. This implies that $G(x^1, x^2, x^3) = (c_1 x^3)^{-2}$ and the third structure equation becomes
$$\mathrm{d}\eta^3 = \eta^1 \wedge \eta^2 - (x^3)^{-1}(2J - x^3 \partial_3 J)\eta^1 \wedge \eta^3 - c_1 \eta^2 \wedge \eta^3.$$
Hence $(x^3)^{-1}(2J - x^3 \partial_3 J) = c_3$, which translates into
$$J(x^1, x^2, x^3) = c_3 x^3 + J_0(x^1, x^2)(x^3)^2.$$
For a function $J_0(x^1, x^2)$. The second structure equation becomes
$$\mathrm{d}\eta^2 = -x^3 \partial_1 J_0 \eta^1 \wedge \eta^3.$$
If $-x^3 \partial_1 J_0$ equals a constant then $\partial_1 J_0 = 0$ hence $J(x^1, x^2, x^3) = c_3 x^3 + J_1(x^2)(x^3)^2$ for a function $J_1(x^2)$. Replacing in (2.18) yields
$$\tilde{J}_1(\tilde{x}^2) = J_1(\phi_0(\tilde{x}^2))\phi_0'(\tilde{x}^2) - \frac{\phi_0''(\tilde{x}^2)}{\phi_0'(\tilde{x}^2)}.$$
Hence the local coordinates can be chosen so that
$$x^1 = \tilde{x}^1 + a, \qquad x^2 = b\tilde{x}^2 + c, \qquad x^3 = b\tilde{x}^3 + c.$$

- **Second Case**: the distribution $\mathcal{F}$ is
$$\mathcal{F} = \left(x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J \frac{\partial}{\partial x^3} \right) + \mathrm{span}\left(\frac{\partial}{\partial x^3} \right),$$
where J is an arbitrary function on M.

This case follows the same steps as the previous one, but this time it is possible to take the structure equations directly from the frame given by the distribution.

The canonical frame is:
$$\begin{aligned} v_1 &= x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + x^2 J \frac{\partial}{\partial x^3}, \\ v_2 &= x^2 \frac{\partial}{\partial x^3}, \\ v_3 &= x^2 \frac{\partial}{\partial x^2} + \left((x^2)^2 \partial_3 J - x^3 \right) \frac{\partial}{\partial x^3}. \end{aligned}$$

The corresponding canonical coframe is:

$$\begin{aligned}
\eta^1 &= \frac{1}{x^2}\mathrm{d}x^1, \\
\eta^2 &= \frac{1}{x^2}\mathrm{d}x^3 - \frac{J}{x^2}\mathrm{d}x^1 - \Big(\partial_3 J - \frac{x^3}{(x^2)^2}\Big)\Big(\mathrm{d}x^2 - \frac{x^3}{x^2}\mathrm{d}x^1\Big), \\
\eta^3 &= \frac{1}{x^2}\mathrm{d}x^2 - \frac{x^3}{(x^2)^2}\mathrm{d}x^1.
\end{aligned}$$

The corresponding structure equations are

$$\begin{aligned}
\mathrm{d}\eta^1 &= \eta^1 \wedge \eta^3, \\
\mathrm{d}\eta^2 &= T^2_{13}\eta^1 \wedge \eta^3 + T^2_{23}\eta^2 \wedge \eta^3, \\
\mathrm{d}\eta^3 &= \eta^1 \wedge \eta^2 + T^3_{13}\eta^1 \wedge \eta^3.
\end{aligned}$$

The allowed transformations are

$$x^1 = \phi(\tilde{x}^1), \qquad x^2 = \phi'(\tilde{x}^1)\tilde{x}^2, \qquad x^3 = \phi'(\tilde{x}^1)\tilde{x}^3 + \phi''(\tilde{x}^1)(\tilde{x}^2)^2$$

where $\phi'(\tilde{x}^1) \neq 0$.
These transformations imply that

$$\tilde{J}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = J(x^1, x^2, x^3) - \frac{1}{\phi'(\tilde{x}^1)}\Big(\phi'''(\tilde{x}^1)(\tilde{x}^2)^2 + 3\phi''(\tilde{x}^1)\tilde{x}^3\Big)$$

The last structure equation gives

$$T^2_{12} = x^2\partial_3 - 3\frac{x^3}{x^2},$$

Making $T^2_{12} = a$ for homogeneity translates into

$$J(x^1, x^2, x^3) = \frac{3}{2}\Big(\frac{x^3}{x^2}\Big)^2 + a\frac{x^3}{x^2} + J_0(x^1, x^2),$$

for some function. Similarly the second structure equation implies that

$$T^2_{13} = x^2\partial_2 J_0 - 2J_0 - a\frac{x^3}{x^2}.$$

In order for T^2_{13} to be a constant we make $a = 0$, resulting in

$$x^2\partial_2 J_0 - 2J_0 = -2c_1,$$

for some constant c_1. Thus,

$$J_0(x^1, x^2) = c_1 + J_1(x^1)(x^2)^2$$

for some function $J_1(x^1)$, and

$$J(x^1, x^2, x^3) = \frac{3}{2}\left(\frac{x^3}{x^2}\right)^2 + c_1 + J_1(x^1)(x^2)^2.$$

At last, replacing in the allowed transformation gives

$$\tilde{J}_1(\tilde{x}^1) = \phi'(\tilde{x}^1)^2 J_1(\phi(\tilde{x}^1)) - \frac{\phi'''(\tilde{x}^1)}{\phi'(\tilde{x}^1)} + \frac{3}{2}\frac{\phi''(\tilde{x}^1)}{(\phi'(\tilde{x}^1))^2}$$

Thus, normalizing $\tilde{J}_1(\tilde{x}^1) = 0$ results in the following differential equation,

$$\frac{\phi'''(\tilde{x}^1)}{\phi'(\tilde{x}^1)} - \frac{3}{2}\frac{\phi''(\tilde{x}^1)}{(\phi'(\tilde{x}^1))^2} = 0,$$

whose solution is,

$$\phi(\tilde{x}^1) = \frac{a\tilde{x}^1 + b}{c\tilde{x}^1 + d},$$

So finally the allowed transformations are

$$x^1 = \frac{a\tilde{x}^1 + b}{c\tilde{x}^1 + d}, \qquad x^2 = \frac{ad - bc}{(c\tilde{x}^1 + d)^2}\tilde{x}^2, \qquad x^3 = \frac{ad - bc}{(c\tilde{x}^1 + d)^2}\tilde{x}^3 - \frac{2c(ad - bc)}{(c\tilde{x}^1 + d)^3}(\tilde{x}^2)^2.$$

- **Third Case**: the distribution $\mathcal{F}$ is

$$\mathcal{F} = \left((1 + x^3 J)\frac{\partial}{\partial x^1} + J\frac{\partial}{\partial x^2} + JH\frac{\partial}{\partial x^3}\right) + \operatorname{span}\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right)$$

where J and H are arbitrary functions on M satisfying $\frac{\partial H}{\partial x^1} \neq 0$. This case follows the same procedure as before but is much more longer, for details see [2].

□

2.2 Rank-2 Distributions in 3-Manifolds

Theorem 2.2.1 (Local classification of rank-2 strictly affine distributions on 3-manifolds)**.** Let $\mathcal{F}$ be a rank 2 strictly affine point-affine distribution of constant type on a manifold M of dimension 3. Then:

1. If $\mathcal{F}$ is almost bracket-generating, then in a sufficiently small neighborhood of any point $x \in M$, there exist local coordinates (x^1, x^2, x^3) such that

$$\mathcal{F} = \left(\frac{\partial}{\partial x^1}\right) + \operatorname{span}\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right).$$

2. If $\mathcal{F}$ is bracket-generating and $L_{\mathcal{F}}$ is completely integrable, then in a sufficiently small neighborhood of any point $x \in M$, there exist local coordinates (x^1, x^2, x^3) such that

$$\mathcal{F} = \left(x^2\frac{\partial}{\partial x^1} - J_1\frac{\partial}{\partial x^2}\right) + \operatorname{span}\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right),$$

   where J_1 is an arbitrary function on M.

3. If $\mathcal{F}$ is bracket-generating and $L_{\mathcal{F}^2}$ is not completely integrable, then in a sufficiently small neighborhood of any point $x \in M$, there exist local coordinates (x^1, x^2, x^3) such that

$$\mathcal{F} = \left((1 + x^3 J_3)\frac{\partial}{\partial x^1} + J_3\frac{\partial}{\partial x^2} - J_2\frac{\partial}{\partial x^3}\right) + \operatorname{span}\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right),$$

   where J_2 and J_3 are arbitrary functions on M.

Proof. The proof follows the same steps as Theorem 2.1.1, namely start with a local framing (v_1, v_2, v_3) on M, the distinguished vector field will be v_1 and the vectors v_2, v_3 will generate the distributions, that is:

$$\mathcal{F} = v_1 + \operatorname{span}(v_2, v_3).$$

In order for another local framing $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ to be equivalent it must generate the same distribution, which implies that

$$\tilde{v}_1 = v_1, \qquad \tilde{v}_2 = b_2 v_2 + c_2 v_3, \qquad \tilde{v}_3 = b_3 v_2 + c_3 v_3,$$

where $b_2 c_3 \neq c_2 b_3$.

These conditions extend to the coframings (η^1, η^2, η^3) and $(\tilde{\eta}^1, \tilde{\eta}^2, \tilde{\eta}^3)$ in M by:

$$\begin{pmatrix} \tilde{\eta}^1 \\ \tilde{\eta}^2 \\ \tilde{\eta}^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{pmatrix}^{-1} \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix}. \tag{2.20}$$

The problem of classifying equivalent coframings in M can be solved using the Cartan reduction method in the lifted space $B(M)$ via the map $g^{-1}\mathrm{d}\pi$ as in (1.1) with the corresponding principal bundle B_0 with structure group G_0 defined by:

$$B_0 = \{(x, \xi_x) \in B(M) \mid x \in \mathbb{R}^3,\ \xi_x \in F_x(T^*\mathbb{R}^3),\ \text{i.e., } \xi_x = (\theta^1_x, \theta^2_x, \theta^3_x)^T\},$$
$$G_0 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{pmatrix} \,\middle|\, b_2c_3 \neq c_2b_3 \right\}.$$

Next, follow the same steps as Theorem 2.1.1 (see section 1.2):

1. Set $i = 0$
2. Find a basis for $\mathfrak{g}_\mathfrak{i}$, the Lie algebra of G_i.
3. Extend to a basis of $\mathfrak{g}_\mathfrak{i} \otimes \mathbb{R}^3$.
4. Using the Spencer operator δ find a base for $\delta(\mathfrak{g}_\mathfrak{i} \otimes \mathbb{R}^3)$.
5. Write a base for $\mathcal{T}_i$ and the structure function $\mathcal{C}$.
6. Compute the orbits of $\mathcal{T}_i$.
7. There are two options:
   (a) If the G_i-action on $\mathcal{T}_i$ is trivial the reduction is finished, and the resulting structure equations give the normal frame and coframe.
   (b) If the G_i-action on $\mathcal{T}_i$ is not trivial, then compute the transformations of the structure functions that preserve a normal form and the G_{i+1}-Structure $B_{i+1} \longrightarrow \mathbb{R}^3$, set $i = i + 1$ and go to step 2.

A different proof is given in [1].

□

Theorem 2.2.2 (Normal form classification of rank-2 strictly affine distributions on 3-manifolds)**.** Let $\mathcal{F}$ be a rank-2, strictly-affine, bracket-generating or almost bracket-generating point-affine distribution of constant type on a 3-dimensional manifold M. If the structure equations of $\mathcal{F}$ are homogeneous, then $\mathcal{F}$ is locally point-affine equivalent to

$$\mathcal{F} = v_1 + \mathrm{span}(v_2, v_3). \tag{2.21}$$

Where the vectors v_1, v_2 and v_3 are one of the following:

- Case 1:
$$v_1(x) = \frac{\partial}{\partial x^1}, \qquad v_2(x) = \frac{\partial}{\partial x^2}, \qquad v_3(x) = \frac{\partial}{\partial x^3}.$$
- Case 2:
$$v_1(x) = x^2\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}, \qquad v_2(x) = x^2\frac{\partial}{\partial x^2}, \qquad v_3(x) = \frac{\partial}{\partial x^3},$$

- Case 3:

$$v_1(x) = (1 + c_3 x^3)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \qquad v_2(x) = x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \qquad v_3(x) = \frac{\partial}{\partial x^3}.$$

Proof. Start analyzing each one of the three cases in Theorem 2.2.1:

- Case 1: the canonical frame corresponding to the associate distribution is

$$v_1 = \frac{\partial}{\partial x^1}, \quad v_2 = \frac{\partial}{\partial x^2}, \quad v_3 = \frac{\partial}{\partial x^3}.$$

  The corresponding coframe is

$$\eta_1 = \mathrm{d}x^1, \quad \eta_2 = \mathrm{d}x^2, \quad \eta_3 = \mathrm{d}x^3.$$

  The corresponding structure equations are trivial

$$\mathrm{d}\eta_1 = 0, \quad \mathrm{d}\eta_2 = 0, \quad \mathrm{d}\eta_3 = 0. \tag{2.22}$$

  Equations (2.22) imply that the distribution $\mathcal{F}$ is homogeneous with structure functions $T^i_{jk} = 0$. If $(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3)$ are the one-forms associated with the coordinate system $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ then the structure equations (2.22) are preserved by transformations of the form

$$x^1 = \tilde{x}^1 + c_1, \quad x^2 = \tilde{x}^2 + c_2, \quad x^3 = \tilde{x}^3 + c_3.$$

- Case 2: In this case proceeding as before and starting with the canonical frame

$$\begin{aligned} v_1 &= x^2\frac{\partial}{\partial x^1} - J_1\frac{\partial}{\partial x^2}, \\ v_2 &= \frac{\partial}{\partial x^2}, \\ v_3 &= \frac{\partial}{\partial x^3}, \end{aligned}$$

  will result in the following coframe

$$\begin{aligned} \eta_1 &= \frac{1}{x^2}\mathrm{d}x^1, \\ \eta_2 &= \frac{J_1}{x^2}\mathrm{d}x^1 + \mathrm{d}x^2, \\ \eta_3 &= \mathrm{d}x^3. \end{aligned}$$

  Computing the wedge products $\eta_1 \wedge \eta_2$, $\eta_1 \wedge \eta_3$ and $\eta_2 \wedge \eta_3$ the first structure equation becomes

$$\mathrm{d}\eta^1 = \frac{1}{x^2}\eta^1 \wedge \eta^2.$$

It is clear that the structure functions cannot be constant, hence it is necessary to consider other frames. The simplest case is

$$
\begin{aligned}
v_1 &= x^2\frac{\partial}{\partial x^1} - J_1\frac{\partial}{\partial x^2},\\
v_2 &= \beta_2^2\frac{\partial}{\partial x^2} + \beta_2^3\frac{\partial}{\partial x^3},\\
v_3 &= \beta_3^2\frac{\partial}{\partial x^2} + \beta_3^3\frac{\partial}{\partial x^3}.
\end{aligned} \tag{2.23}
$$

Where in principle β_2^2, β_3^2, β_2^3 and β_3^3 are functions of x^1, x^2 and x^3 such that v_2 and v_3 remain linearly independent. The corresponding coframe is

$$
\begin{aligned}
\eta^1 &= \frac{1}{x^2}\mathrm{d}x^1,\\
\eta^2 &= \frac{J_1\beta_3^3}{x^2\Delta\beta}\mathrm{d}x^1 + \frac{\beta_3^3}{\Delta\beta}\mathrm{d}x^2 - \frac{\beta_3^2}{\Delta\beta}\mathrm{d}x^3,\\
\eta^3 &= -\frac{J_1\beta_2^3}{x^2\Delta\beta}\mathrm{d}x^1 - \frac{\beta_2^3}{\Delta\beta}\mathrm{d}x^2 - \frac{\beta_2^2}{\Delta\beta}\mathrm{d}x^3,
\end{aligned}
$$

where $\Delta\beta = \beta_3^3\beta_2^2 - \beta_3^2\beta_2^3$. Now, since β_i^j are functions of x^1, x^2 and x^3, calculating directly the differentials of the coframe one-forms is too long. Instead, first calculate the wedge products

$$
\begin{aligned}
\eta^1\wedge\eta^2 &= \frac{\beta_3^3}{x^2\Delta\beta}\mathrm{d}x^1\wedge\mathrm{d}x^2 - \frac{\beta_3^2}{x^2\Delta\beta}\mathrm{d}x^1\wedge\mathrm{d}x^3,\\
\eta^1\wedge\eta^3 &= \frac{-\beta_2^3}{x^2\Delta\beta}\mathrm{d}x^1\wedge\mathrm{d}x^2 + \frac{\beta_2^2}{x^2\Delta\beta}\mathrm{d}x^1\wedge\mathrm{d}x^3,\\
\eta^2\wedge\eta^3 &= \frac{J_1}{x^2\Delta\beta}\mathrm{d}x^1\wedge\mathrm{d}x^3 + \frac{1}{\Delta\beta}\mathrm{d}x^2\wedge\mathrm{d}x^3.
\end{aligned}
$$

Consider a general structure equation

$$
\begin{aligned}
\mathrm{d}\eta^i &= \gamma_{12}^i\mathrm{d}x^1\wedge\mathrm{d}x^2 + \gamma_{13}^i\mathrm{d}x^1\wedge\mathrm{d}x^3 + \gamma_{23}^i\mathrm{d}x^2\wedge\mathrm{d}x^3,\\
&= T_{12}^i\eta^1\wedge\eta^2 + T_{13}^i\eta^1\wedge\eta^3 + T_{23}^i\eta^2\wedge\eta^3.
\end{aligned} \tag{2.24}
$$

Replacing the wedge products $\eta^1\wedge\eta^2$, $\eta^1\wedge\eta^3$ and $\eta^2\wedge\eta^3$ the following equations are obtained by comparing the coefficients:

$$
\begin{aligned}
T_{12}^i &= x^2(\beta_2^2\gamma_{12}^i + \beta_2^3\gamma_{13}^i) - J_1\beta_2^3\gamma_{23}^i,\\
T_{13}^i &= x^2(\beta_3^2\gamma_{12}^i + \beta_3^3\gamma_{13}^i) - J_1\beta_3^3\gamma_{23}^i,\\
T_{23}^i &= \gamma_{23}^i\Delta\beta.
\end{aligned}
$$

For $i = 1$

$$
\mathrm{d}\eta^1 = \frac{-1}{(x^2)^2}\mathrm{d}x^1\wedge\mathrm{d}x^2 \quad\Longrightarrow\quad \gamma_{12}^1 = \frac{1}{(x^2)^2}, \qquad \gamma_{13}^1 = 0, \qquad \gamma_{23}^1 = 0.
$$

Hence

$$T^1_{12} = x^2\beta^2_2\frac{1}{(x^2)^2}, \qquad T^1_{13} = x^2\beta^2_3\frac{1}{(x^2)^2}, \qquad T^1_{23} = 0. \tag{2.25}$$

In order for the structure to be homogeneous the structure functions must be constant. Let $T^1_{12} = \tau^1_{12}$, and $T^1_{13} = \tau^1_{13}$, where τ^1_{12} and τ^1_{13} are constants, then

$$\beta^2_2 = x^2\tau^1_{12}, \qquad \beta^2_3 = x^2\tau^1_{13}. \tag{2.26}$$

Although the functions β^2_2 and β^2_3 are known, the direct calculation of $\mathrm{d}\eta^2$ and $\mathrm{d}\eta^3$ is still very long given that β^3_3 and β^3_2 are arbitrary functions of x^1, x^2 and x^3. In order to simplify the calculation assume that β^3_3 and β^3_2 are constants $\beta^3_3 = b^3_3$ and $\beta^3_2 = b^3_2$. In this case

$$\Delta\beta = x^2(\tau^1_{12}b^3_3 - \tau^1_{13}b^3_2) = x^2 b$$

Hence the one-forms η^2 and η^3 become

$$\eta^2 = \frac{J_1 b^3_3}{(x^2)^2 b}\mathrm{d}x^1 + \frac{b^3_3}{x^2 b}\mathrm{d}x^2 - \frac{\tau^1_{13}}{b}\mathrm{d}x^3,$$
$$\eta^3 = \frac{-J_1 b^3_2}{(x^2)^2 b}\mathrm{d}x^1 - \frac{b^3_2}{x^2 b}\mathrm{d}x^2 + \frac{\tau^1_{12}}{b}\mathrm{d}x^3,$$

and

$$\mathrm{d}\eta^2 = \frac{-b^3_3}{b}\partial_2\Big(\frac{J_1}{(x^2)^2}\Big)\mathrm{d}x^1\wedge\mathrm{d}x^2 - \frac{b^3_3}{(x^2)^2 b}(\partial_3 J_1)\mathrm{d}x^1\wedge\mathrm{d}x^3,$$
$$\mathrm{d}\eta^3 = \frac{b^3_2}{b}\partial_2\Big(\frac{J_1}{(x^2)^2}\Big)\mathrm{d}x^1\wedge\mathrm{d}x^2 + \frac{b^3_2}{(x^2)^2 b}(\partial_3 J_1)\mathrm{d}x^1\wedge\mathrm{d}x^3.$$

Thus from (2.24)

$$\gamma^2_{12} = \frac{-b^3_3}{b}\partial_2\Big(\frac{J_1}{(x^2)^2}\Big), \qquad \gamma^2_{13} = \frac{-b^3_3}{(x^2)^2 b}(\partial_3 J_1), \qquad \gamma^2_{23} = 0,$$
$$\gamma^3_{12} = \frac{b^3_2}{b}\partial_2\Big(\frac{J_1}{(x^2)^2}\Big), \qquad \gamma^3_{13} = \frac{b^3_2}{(x^2)^2 b}(\partial_3 J_1), \qquad \gamma^3_{23} = 0.$$

Replacing in the structure functions

$$T^2_{12} = -x^2\left(x^2\tau^1_{12}\frac{b^3_3}{b}\partial_2\Big(\frac{J_1}{(x^2)^2}\Big) + b^3_2\frac{b^3_3}{(x^2)^2 b}(\partial_3 J_1)\right), \tag{2.27}$$

$$T^2_{13} = -x^2\left(x^2\tau^1_{13}\frac{b^3_3}{b}\partial_2\Big(\frac{J_1}{(x^2)^2}\Big) + b^3_3\frac{b^3_3}{(x^2)^2 b}(\partial_3 J_1)\right), \tag{2.28}$$

$$T^2_{23} = 0, \tag{2.29}$$

$$T^3_{12} = x^2\left(x^2\tau^1_{12}\frac{b^3_2}{b}\partial_2\Big(\frac{J_1}{(x^2)^2}\Big) + b^3_2\frac{b^3_2}{(x^2)^2 b}(\partial_3 J_1)\right), \tag{2.30}$$

$$T^3_{13} = x^2\left(x^2\tau^1_{13}\frac{b^3_2}{b}\partial_2\Big(\frac{J_1}{(x^2)^2}\Big) + b^3_3\frac{b^3_2}{(x^2)^2 b}(\partial_3 J_1)\right), \tag{2.31}$$

$$T^3_{23} = 0. \tag{2.32}$$

Thanks to the symmetry of the equations, the structure functions T^2_{12}, T^2_{13}, T^3_{12} and T^3_{13} will be equal to constants if

$$(x^2)^2\kappa_l\partial_2\Big(\frac{J_1}{(x^2)^2}\Big)+\frac{\chi_l}{x^2}\partial_3 J_1=c_l, \tag{2.33}$$

where κ_l, χ_l and c_l are constants that satisfy:

1. For $l=1$, $\kappa_1=-\tau^1_{12}b^3_3(b^{-1})$, $\chi_1=b^3_2b^3_3(b^{-1})$, hence replacing (2.33) into (2.27) implies $T^2_{12}=c_1$.
2. For $l=2$, $\kappa_2=-\tau^1_{13}b^3_3(b^{-1})$, $\chi_2=b^3_3b^3_3(b^{-1})$, hence replacing (2.33) into (2.28) implies $T^2_{13}=c_2$.
3. For $l=3$, $\kappa_3=\tau^1_{12}b^3_2(b^{-1})$, $\chi_3=b^3_2b^3_2(b^{-1})$, hence replacing (2.33) into (2.30) implies $T^3_{12}=c_3$.
4. For $l=4$, $\kappa_4=\tau^1_{13}b^3_2(b^{-1})$, $\chi_4=b^3_3b^3_2(b^{-1})$, hence replacing (2.33) into (2.31) implies $T^3_{13}=c_4$.

Equation (2.33) can be written as

$$\kappa_l\partial_2 J_1-\frac{2J_1\kappa_l}{x^2}+\frac{\chi_l\partial_3 J_1}{x^2}=c_l.$$

Since it is a sum of three terms if each one is constant then the sum will be constant. If the second term of the left hand side is constant then

$$J_1=x^2\widehat{J}_1 \quad\Longrightarrow\quad \partial_2 J_1=\widehat{J}_1, \qquad \partial_3 J_1=0,$$

and the resulting structure equations become homogeneous, as desired.

In summary if $J_1=x^2\widehat{J}_1$ then (2.33) is satisfied and the structure functions (2.27), (2.28), (2.30) and (2.31) become constants. On the other hand (2.26) imply that the structure functions on (2.25) are constants and since the structure functions (2.29) and (2.32) are already constants the structure equations (2.24) become homogeneous.

Now, normalize the constants $\widehat{J}_1=-1$, $\tau^1_{12}=1$, $\tau^1_{13}=0$, $\tau^3_2=0$ and $\tau^3_3=1$ the final frame is

$$\begin{aligned}
v_1&=x^2\frac{\partial}{\partial x^1}+x^2\frac{\partial}{\partial x^2},\\
v_2&=x^2\frac{\partial}{\partial x^2},\\
v_3&=\frac{\partial}{\partial x^3}.
\end{aligned}$$

The corresponding coframe is

$$\begin{aligned}
\eta^1&=\frac{1}{x^2}\mathrm{d}x^1,\\
\eta^2&=\frac{-1}{x^2}\mathrm{d}x^1+\frac{1}{x^2}\mathrm{d}x^2,\\
\eta^3&=\mathrm{d}x^3,
\end{aligned}$$

the associated structure functions are

$$\begin{aligned} T^1_{12} &= 1, \qquad & T^1_{13} &= 0, \qquad & T^1_{23} &= 0, \\ T^2_{12} &= -1, \qquad & T^2_{13} &= 0, \qquad & T^2_{23} &= 0, \\ T^3_{12} &= 0, \qquad & T^3_{13} &= 0, \qquad & T^3_{23} &= 0, \end{aligned}$$

hence the structure equations are homogeneous as desired. Finally in order to calculate the coordinate transformations that preserve the coframe, let $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ be another coordinate system related to (x^1, x^2, x^3) by

$$\begin{aligned} x^1 &= \phi_1(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \\ x^2 &= \phi_2(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \\ x^3 &= \phi_3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3). \end{aligned}$$

Making $\eta^1 = \tilde{\eta}^1$ results in

$$x^1 = \phi_1(\tilde{x}^1), \qquad x^2 = \tilde{x}^2 \partial_1 \phi_1(\tilde{x}^1).$$

Replacing in η^2 and using $\eta^2 = \tilde{\eta}^2$ it is obtained

$$\partial_{11}\phi_1 = 0, \qquad \partial_2\phi_3 = 0, \qquad \partial_3\phi_3 = 1,$$

which imply that $\partial_1\phi_1 = K_1 \neq 0$. Finally $\eta^3 = \tilde{\eta}^3$ gives

$$(\partial_1\phi_3)(\partial_1\phi_1) = 0,$$

and at last the allowed transformations are of the form

$$x^1 = K_1\tilde{x}^1, \qquad x^2 = K_1\tilde{x}^2, \qquad x^3 = \tilde{x}^3 + K_2.$$

- Case 3: Start with the canonical frame

$$\begin{aligned} v_1 &= (1 + x^3)\frac{\partial}{\partial x^1} + J_3\frac{\partial}{\partial x^2} - J_2\frac{\partial}{\partial x^3}, \\ v_2 &= x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \\ v_3 &= \frac{\partial}{\partial x^3}. \end{aligned}$$

The corresponding coframe is

$$\begin{aligned} \eta_1 &= \mathrm{d}x^1 - x^3\mathrm{d}x^2, \\ \eta_2 &= -J_3\mathrm{d}x^1 + (1 + J_3x^3)\mathrm{d}x^2, \\ \eta_3 &= J_2\mathrm{d}x^1 - x^3J_2\mathrm{d}x^2 + \mathrm{d}x^3. \end{aligned}$$

Computing the structure equations yield the following structure functions:

$$
\begin{aligned}
T^1_{12} &= J_2,\\
T^1_{13} &= J_3,\\
T^1_{23} &= 1,\\
T^2_{12} &= ((\partial_2 J_3) + x^3(\partial_1 J_3)) - J_2((\partial_3 J_3)(1+x^3) + J_3),\\
T^2_{13} &= -J_3((\partial_3 J_3)(1+x^3)) - J_3^2,\\
T^2_{23} &= -((\partial_3 J_3)(1+x^3) + J_3),\\
T^3_{12} &= J_2^2 + x^3 J_2(\partial_3 J_2) - ((\partial_2 J_2) + x^3(\partial_1 J_2)),\\
T^3_{13} &= J_2 J_3 - x^3(\partial_3 J_2) J_3 - (\partial_3 J_3),\\
T^3_{23} &= J_2 + x^3(\partial_3 J_2).
\end{aligned}
$$

The first two equations simplify the analysis, since they imply that $J_2 = c_2$ and $J_3 = c_3$, which consequently results in the following structure equations

$$
\begin{aligned}
\mathrm{d}\eta^1 &= c_2\eta^1 \wedge \eta^2 + c_3\eta^1 \wedge \eta^3 + \eta^2 \wedge \eta^3,\\
\mathrm{d}\eta^2 &= -c_2c_3\eta^1 \wedge \eta^2 - c_3^2\eta^1 \wedge \eta^3 - c_3\eta^2 \wedge \eta^3,\\
\mathrm{d}\eta^3 &= c_2^2\eta^1 \wedge \eta^2 + c_1c_2\eta^1 \wedge \eta^3 + c_2\eta^2 \wedge \eta^3.
\end{aligned}
$$

Thus the structure is homogeneous and the normal form is obtained with $c_2 = -1$ and $c_3 = 1$.

The next step is to determine the coordinate transformations that preserve the coframe. If $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ is another coordinate system related with (x^1, x^2, x^3) by the following equations

$$
x^1 = \phi_1(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \qquad x^2 = \phi_2(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \qquad x^3 = \phi_3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3).
$$

Then each one-form gives three equations that must be satisfied:

$$
\eta^1 = \tilde{\eta}^1 \Longrightarrow \begin{cases} \partial_1\phi_1 &= \phi_3\partial_1\phi_2 + 1,\\ \partial_2\phi_1 &= \phi_3\partial_2\phi_2 - \tilde{x}^3,\\ \partial_3\phi_1 &= \phi_3\partial_3\phi_2, \end{cases} \tag{2.34}
$$

$$
\eta^2 = \tilde{\eta}^2 \Longrightarrow \begin{cases} (1+c_3\phi_3)\partial_1\phi_2 &= c_3\partial_1\phi_1 - c_3,\\ (1+c_3\phi_3)\partial_2\phi_2 &= (1+c_3\tilde{x}^3) + c_3\partial_2\phi_1,\\ (1+c_3\phi_3)\partial_3\phi_2 &= c_3\partial_3\phi_1, \end{cases} \tag{2.35}
$$

$$
\eta^3 = \tilde{\eta}^3 \Longrightarrow \begin{cases} \partial_1\phi_1 &= \phi_3\partial_1\phi_2 + (\partial_1\phi_3)(c_2)^{-1},\\ \partial_2\phi_1 &= -\tilde{x}^3 + \phi_3\partial_2\phi_2 - (\partial_2\phi_3)(c_2)^{-1},\\ \partial_3\phi_1 &= (c_2)^{-1} + \phi_3\partial_3\phi_2 - (\partial_3\phi_3)(c_2)^{-1}. \end{cases} \tag{2.36}
$$

Comparing equations 2.34 and 2.36 results in the following equations:

$$
\partial_1\phi_3 = c_2, \qquad \partial_2\phi_3 = 0, \qquad \partial_3\phi_3 = 1.
$$

Hence

$$x^3 = \phi_3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \tilde{x}^3 + c_2\tilde{x}^2.$$

Replacing this result into (2.34) and (2.35) yield the following equations

$$\partial_1\phi_1 = 1, \qquad \partial_2\phi_1 = c_2\tilde{x}^2, \qquad \partial_3\phi_1 = 0,$$

$$\partial_1\phi_2 = 0, \qquad \partial_2\phi_2 = 1, \qquad \partial_3\phi_2 = 0,$$

which in turn imply that the complete allowed transformations are:

$$x^1 = \tilde{x}^1 + \frac{1}{2}c_2(\tilde{x}^2)^2 + a_1, \qquad x^2 = \tilde{x}^2 + a_2, \qquad x^3 = \tilde{x}^3 + c_2\tilde{x}^2.$$

Normalize $c_2 = 1$ and $c_3 = 1$ to obtain the normal frame

$$\begin{aligned}
v_1 &= (1 + x^3)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}, \\
v_2 &= x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \\
v_3 &= \frac{\partial}{\partial x^3}.
\end{aligned}$$

The allowed transformations are

$$x^1 = \tilde{x}^1 + \frac{1}{2}(\tilde{x}^2)^2 + a_1, \qquad x^2 = \tilde{x}^2 + a_2, \qquad x^3 = \tilde{x}^3 + \tilde{x}^2.$$

$\square$

Chapter 3

Examples

3.1 Three States, One Control

Consider the following dynamical system:

$$\begin{aligned}
\dot{r} &= \cos\theta + z\sin\theta, \\
\dot{\theta} &= \frac{1}{r}\big(z\cos\theta - \sin\theta\big), \\
\dot{z} &= c_2 r\sin\theta + c_3 z + u.
\end{aligned}$$

Where the state vector $x = (r, \theta, z)$ takes values in the state space $M = [0, \infty) \times [0, 2\pi) \times \mathbb{R}$. We would like to know if this dynamic system corresponds to one of the cases of theorem 2.1.2. First of all it is necessary to check that the associated affine distribution

$$\mathcal{F} = \left(\big(\cos\theta + z\sin\theta\big)\frac{\partial}{\partial r} + \frac{1}{r}\big(z\cos\theta - \sin\theta\big)\frac{\partial}{\partial\theta} + \big(c_2 r\sin\theta + c_3 z\big)\frac{\partial}{\partial z}\right) + \operatorname{span}\left(\frac{\partial}{\partial z}\right),$$

satisfies all the hypothesis of the theorem:

1. The distribution $\mathcal{F}$ is strictly affine: Suppose it is not, then there exists a state $\bar{x} = (\bar{r}, \bar{\theta}, \bar{z})$ such that $v_1(\bar{x}) = 0$ this would imply

$$\cos\bar{\theta} + \bar{z}\sin\bar{\theta} = 0, \tag{3.1}$$

$$\frac{1}{\bar{r}}\big(\bar{z}\cos\bar{\theta} - \sin\bar{\theta}\big) = 0, \tag{3.2}$$

$$c_2\bar{r}\sin\bar{\theta} + c_3\bar{z} = 0. \tag{3.3}$$

From (3.2)

$$\bar{z}\cos\bar{\theta} = \sin\bar{\theta}, \tag{3.4}$$

replacing in (3.1) it is obtained

$$\cos\bar{\theta}(1 + \bar{z}^2) = 0,$$

which can only be satisfied if $\cos\bar{\theta} = 0$. Replacing in (3.4) also implies that $\sin\bar{\theta} = 0$, but there is no $\bar{\theta}$ that can satisfy both equations, hence the distribution is strictly affine.

2. The distribution $\mathcal{F}$ is bracket-generating or almost bracket-generating of constant type: We start with the original distribution $\mathcal{F}$ and from then construct the flag of subsheaves

$$\mathcal{F} = \mathcal{F}^1 \subset \mathcal{F}^2 \subset \cdots \subset TM,$$

using the definition $\mathcal{F}^{i+1} = \mathcal{F}^i + [\mathcal{F}, \mathcal{F}^i]$.

(a) $\mathcal{F}^1 = v_1 + \operatorname{span}(v_2)$.

(b) $\mathcal{F}^2 = \mathcal{F}^1 + [\mathcal{F}, \mathcal{F}^1]$, where

$$[\mathcal{F}, \mathcal{F}^1] = [v_1 + \lambda_2 v_2, v_1 + \beta_2 v_2] = \underbrace{[v_1, v_1]}_{0} + (\beta_2 - \lambda_2) \underbrace{[v_1, v_2]}_{v_3} + \beta_2 \lambda_2 \underbrace{[v_2, v_2]}_{0},$$

thus the only new vector is $v_3 = [v_1, v_2]$:

$$v_3 = \sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta} + c_3 \frac{\partial}{\partial z},$$

and $\mathcal{F}^2 = v_1 + \operatorname{span}(v_2, v_3)$.

(c) $\mathcal{F}^3 = \mathcal{F}^2 + [\mathcal{F}, \mathcal{F}^2]$, where

$$\begin{aligned} [\mathcal{F}, \mathcal{F}^2] &= [v_1 + \lambda_2 v_2, v_1 + \beta_2 v_2 + \beta_3 v_3] \\ &= \underbrace{[v_1, v_1]}_{0} + (\beta_2 - \lambda_2) \underbrace{[v_1, v_2]}_{v_3} + \beta_3 \underbrace{[v_1, v_3]}_{v_4} + \beta_2 \lambda_2 \underbrace{[v_2, v_2]}_{0} \lambda_2 \beta_3 \underbrace{[v_2, v_3]}_{v_5}, \end{aligned}$$

hence the new vectors are $v_4 = [v_1, v_3]$ and $v_5 = [v_2, v_3]$:

$$\begin{aligned} v_4 &= c_3 \sin\theta \frac{\partial}{\partial r} + \frac{c_3}{r}\cos\theta \frac{\partial}{\partial \theta} + (c_2 + c_3^2) \frac{\partial}{\partial z}, \\ v_5 &= 0. \end{aligned}$$

Note that $v_4 = c_3 v_3 + c_2 v_2$, hence $\mathcal{F}^\infty = \cup_{i \geq 1} \mathcal{F}^i = \mathcal{F}^2$ since taking further iterations will not yield new linearly independent vector fields:

$$\begin{aligned} v_6 &= [v_1, v_4] = c_3 [v_1, v_3] + c_2 [v_1, v_2] = c_1 v_4 + c_2 v_3, \\ v_7 &= [v_2, v_4] = c_3 [v_2, v_3] + c_2 [v_2, v_2] = 0, \\ v_8 &= [v_3, v_4] = c_3 [v_3, v_3] + c_2 [v_3, v_2] = 0, \end{aligned}$$

From the above calculations it follows that the step of the distribution $\mathcal{F}$ is 2 and the growth vector is $(1, 2)$.

Since $\dim(\mathcal{F}^\infty) = 2$ and $\dim(TM) = 3$ the distribution is not bracket-generating, but it may be almost bracket-generating. For this it is necessary to check that for each $x \in M$ and any $\xi(x) \in \mathcal{F}_x$, $\operatorname{span}(\xi(x), (L_{\mathcal{F}^\infty})_x) = T_x M = \mathbb{R}^3$.

First compute the direction distribution of $\mathcal{F}^\infty$:

$$L_{\mathcal{F}^\infty} = \{\xi_1 - \xi_2 \mid \xi_1, \xi_2 \in \mathcal{F}^\infty\}.$$

Taking into account that $\mathcal{F}^\infty = v_1 + \lambda_2 v_2 + \lambda_3 v_3$ it follows

$$L_{\mathcal{F}^\infty} = \{v_1 + \beta_2 v_2 + \beta_3 v_3 \mid \beta_2, \beta_3 \in \mathbb{R}\} \sqcup \{\alpha_2 v_2 + \alpha_3 v_3 \mid \alpha_2, \alpha_3 \in \mathbb{R}\},$$

Now, for each $x \in M$, vector field $\xi(x) \in \mathcal{F}_x$ can be written as $\xi(x) = v_1(x) + \lambda_2 v_2(x)$ and since the vector field v_1 is linearly independent from v_2 and v_3 which are contained in $L_{\mathcal{F}^\infty}$ it follows that $\dim(\mathrm{span}(\xi(x), (L_{\mathcal{F}^\infty}))) = 3$, implying that the distribution $\mathcal{F}$ is almost bracket generating.

Finally since the growth vector is constant for any $x \in M$, and $\dim(\mathrm{span}(\xi(x), (L_{\mathcal{F}^\infty})_x))$ is also constant for all $x \in M$ the distribution $\mathcal{F}$ is of constant type.

3. The distribution $\mathcal{F}$ is homogeneous. As usual, first consider the frame

$$\begin{aligned}
v_1 &= \big(\cos\theta + z\sin\theta\big)\frac{\partial}{\partial r} + \frac{1}{r}\big(z\cos\theta - \sin\theta\big)\frac{\partial}{\partial \theta} + \big(c_2 r\sin\theta + c_3 z\big)\frac{\partial}{\partial z},\\
v_2 &= \frac{\partial}{\partial z},\\
v_3 &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} + c_3\frac{\partial}{\partial z},
\end{aligned}$$

which yields the coframe

$$\begin{aligned}
\eta^1 &= \cos\theta \mathrm{d}r - r\sin\theta \mathrm{d}\theta,\\
\eta^2 &= -\big(c_2 r\sin\theta\cos\theta + c_3\sin\theta\big)\mathrm{d}r + r\big(c_2 r\sin^2(\theta) - c_3\cos\theta\big)\mathrm{d}\theta + \mathrm{d}z,\\
\eta^3 &= \big(\sin\theta - z\cos\theta\big)\mathrm{d}r + r\big(\cos\theta + z\sin\theta\big)\mathrm{d}\theta.
\end{aligned}$$

Next compute the derivatives:

$$\begin{aligned}
\mathrm{d}\eta^1 &= 0,\\
\mathrm{d}\eta^2 &= c_2 r\mathrm{d}r \wedge \mathrm{d}\theta,\\
\mathrm{d}\eta^3 &= \cos\theta\mathrm{d}r \wedge \mathrm{d}z - r\sin\theta\mathrm{d}\theta \wedge \mathrm{d}z,
\end{aligned}$$

then we compute the wedge products

$$\begin{aligned}
\eta^1 \wedge \eta^2 &= -c_3 r\mathrm{d}r \wedge \mathrm{d}\theta + \cos\theta\mathrm{d}r \wedge \mathrm{d}z - r\sin\theta\mathrm{d}\theta \wedge \mathrm{d}z\\
\eta^1 \wedge \eta^3 &= r\mathrm{d}r \wedge \mathrm{d}\theta,\\
\eta^2 \wedge \eta^3 &= -r(c_3 z + c_2 r\sin\theta)\mathrm{d}r \wedge \mathrm{d}\theta + (z\cos\theta - \sin\theta)\mathrm{d}r \wedge \mathrm{d}z - r(\cos\theta + z\sin\theta)\mathrm{d}\theta \wedge \mathrm{d}z.
\end{aligned}$$

Finally the structure equations become

$$\begin{aligned}
&T^1_{12} = 0, \quad T^1_{13} = 0, \quad T^1_{23} = 0,\\
&T^2_{12} = 0, \quad T^2_{13} = c_2, \quad T^2_{23} = 0,\\
&T^3_{12} = 1, \quad T^3_{13} = c_3, \quad T^3_{23} = 0,
\end{aligned}$$

hence the structure equations are homogenenous.

Given that the distribution $\mathcal{F}$ satisfies the hypothesis of Theorem 2.1.2 it must be point-affine equivalent to one of the normal forms. In order to find the point affine equivalence it is convenient first to rename the variables.

The original variables (r, θ, z) will be renamed (y^1, y^2, y^3), the state space manifold $\mathcal{Y}$, the distribution $\mathcal{F}_{\mathcal{Y}}$ and the vector fields w_1, w_2, w_3. On the other hand the normal variables will be noted as usual (x^1, x^2, x^3), the normal manifold $\mathcal{X}$, the normal distribution $\mathcal{F}_{\mathcal{X}}$ and the corresponding vector fields v_1, v_2, v_3. With the preceding notation the distributions $\mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}_{\mathcal{Y}}$ are point affine equivalent if there exists a diffeomorphism $\phi : \mathcal{X} \to \mathcal{Y}$ such that

$$\phi_*(v_1(x)) = w_1(\phi(x)), \qquad \phi_*(v_2(x)) = \lambda_2^2(x) w_2(\phi(x)). \tag{3.5}$$

Let

$$v_1 = f_1^1 \tfrac{\partial}{\partial x^1} + f_1^2 \tfrac{\partial}{\partial x^2} + f_1^3 \tfrac{\partial}{\partial x^3}, \qquad w_1 = g_1^1 \tfrac{\partial}{\partial y^1} + g_1^2 \tfrac{\partial}{\partial y^2} + g_1^3 \tfrac{\partial}{\partial y^3},$$

$$v_2 = f_2^1 \tfrac{\partial}{\partial x^1} + f_2^2 \tfrac{\partial}{\partial x^2} + f_2^3 \tfrac{\partial}{\partial x^3}, \qquad w_2 = g_2^1 \tfrac{\partial}{\partial y^1} + g_2^2 \tfrac{\partial}{\partial y^2} + g_2^3 \tfrac{\partial}{\partial y^3},$$

$$v_3 = f_3^1 \tfrac{\partial}{\partial x^1} + f_3^2 \tfrac{\partial}{\partial x^2} + f_3^3 \tfrac{\partial}{\partial x^3}, \qquad w_3 = g_3^1 \tfrac{\partial}{\partial y^1} + g_3^2 \tfrac{\partial}{\partial y^2} + g_3^3 \tfrac{\partial}{\partial y^3},$$

Be the frames in $\mathcal{X}$ and $\mathcal{Y}$, then (3.5) become

$$\sum_{i=1}^{3} \frac{\partial \phi^j}{\partial x^i}(x) f_1^j(x) = g_1^j(\phi(x)), \qquad j = 1, 2, 3$$

$$\sum_{i=1}^{3} \frac{\partial \phi^j}{\partial x^i}(x) f_2^j(x) = \lambda_2^2(x) g_2^j(\phi(x)), \qquad j = 1, 2, 3 \tag{3.6}$$

Given that the distribution is almost bracket-generating Theorem 2.1.2 implies that the dynamical system is point-affine equivalent to case 1 or case 2. Start with case 1, using the new notation for the distribution $\mathcal{F}_{\mathcal{Y}}$ the vector fields become:

$$\begin{aligned}
v_1 &= \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + (c_2 x^2 + c_3 x^3) \frac{\partial}{\partial x^3}, \\
v_2 &= \frac{\partial}{\partial x^3}, \\
w_1 &= \big(\cos(y^2) + y^3 \sin(y^2)\big) \frac{\partial}{\partial y^1} + \frac{1}{y^1}\big(y^3 \cos(y^2) - \sin(y^2)\big) \frac{\partial}{\partial y^2} + \big(c_2 y^1 \sin(y^2) + c_3 y^3\big) \frac{\partial}{\partial y^3}, \\
w_2 &= \frac{\partial}{\partial y^3}.
\end{aligned}$$

Replacing in (3.5) results in the following system of partial differential equations

$$\begin{aligned}
\frac{\partial \phi^1}{\partial x^1} + \frac{\partial \phi^1}{\partial x^2} x^3 + \frac{\partial \phi^1}{\partial x^3}(c_2 x^2 + c_3 x^3) &= \cos(\phi^2) + \psi^3 \sin(\phi^2), \\
\frac{\partial \phi^2}{\partial x^1} + \frac{\partial \phi^2}{\partial x^2} x^3 + \frac{\partial \phi^2}{\partial x^3}(c_2 x^2 + c_3 x^3) &= \frac{1}{\psi^1}\big(\phi^3 \cos(\phi^2) - \sin(\phi^2)\big), \\
\frac{\partial \phi^3}{\partial x^1} + \frac{\partial \phi^3}{\partial x^2} x^3 + \frac{\partial \phi^3}{\partial x^3}(c_2 x^2 + c_3 x^3) &= c_2 \phi^1 \sin(\phi^2) + c_3 \phi^3, \\
\frac{\partial \phi^1}{\partial x^3} &= 0, \\
\frac{\partial \phi^2}{\partial x^3} &= 0, \\
\frac{\partial \phi^3}{\partial x^3} &= \lambda_2^2(x).
\end{aligned}$$

Luckily, years of mathematical education and training allow us to see through this seemingly difficult system of partial differential equations the obvious truth; which is that the coordinate transformation

$$\begin{aligned}
\phi : \mathcal{X} &\longrightarrow \mathcal{Y} \\
(x^1, x^2, x^3) &\longmapsto \begin{pmatrix} \sqrt{(x^1)^2 + (x^2)^2} \\ \tan^{-1}\left(\frac{x^2}{x^1}\right) \\ x^3 \end{pmatrix},
\end{aligned}$$

will solve the system of partial differential equations. Implying that the dynamical system

$$\begin{aligned}
\dot{r} &= \cos\theta + z \sin\theta, \\
\dot{\theta} &= \frac{1}{r}\big(z\cos\theta - \sin\theta\big), \\
\dot{z} &= c_2 r \sin\theta + c_3 z + u.
\end{aligned}$$

It is point-affine equivalent to the system

$$\begin{aligned}
\dot{x}^1 &= 1, \\
\dot{x}^2 &= x^3, \\
\dot{x}^3 &= c_2 x^2 + c_3 x^3 + u.
\end{aligned}$$

3.2 Magnetic Levitator

A magnetic levitator is a system that uses the magnetic force to suspend a metallic ball. A current $i_c(t)$ is passed though an electromagnet generating a magnetic field B that acts on the steel ball generating a magnetic force F_c. This magnetic force acts on the steel ball opposing the weight. The net force moves the ball in the vertical position [9].

The system can be divided into two parts, the first one is the electromagnet that can be modeled as a first order LR circuit

$$v_c(t) = (R_c + R_s)i_c(t) + L_c\frac{\mathrm{d}}{\mathrm{d}t}i_c(t), \tag{3.7}$$

where $v_c(t)$ is the input voltaje, R_c is the coil resistance, R_s is the series resistance and L_c is the coil inductance. The magnetic force acting on the ball is

$$F_c = \frac{K_m i_c(t)^2}{2x_b^2},$$

where K_m is a positive constant and x_b is the distance from the ball to the electromagnet. The total vertical force acting on the ball is

$$F_t = -F_c + F_g = -\frac{K_m i_c(t)^2}{2x_b^2} + gM_b,$$

and by Newton's second law it is equal to the mass of the ball M_b times the acceleration $\ddot{x}_b$, thus

$$\ddot{x}_b(t) = -\frac{K_m i_c(t)^2}{2M_b x_b(t)^2} + g. \tag{3.8}$$

Combining (3.7) and (3.8) and making $\dot{x}_b = v_b$ the following system is obtained:

$$\begin{aligned}
\frac{di_c}{dt} &= -i_c\Big(\frac{R}{L_c}\Big) + v_c\Big(\frac{1}{L_c}\Big),\\
\frac{dv_b}{dt} &= -\frac{K_m i_c^2}{2M_b x_b^2} + g,\\
\frac{dx_b}{dt} &= v_b,
\end{aligned}$$

with state variables $i_c(t)$ the current in the coil, $v_b(t)$ the vertical velocity of the ball and $x_b(t)$ the vertical position of the ball. The input of the system is the voltage of the coil $v_c(t)$, and the parameters are $R = R_c + R_s$, the sum of the coil resistance R_c and the series resistance R_s, L_c the coil inductance, M_b is the mass of the ball, and g the gravity.

The state variables $i_c(t)$, $v_b(t)$ and $x_b(t)$ form the state vector $x(t)$. Notice that there is a singularity if $x_b = 0$, this happens because the magnetic force is inverse proportional to the square of the distance between the ball and the coil, in order to avoid this problem the state variable x_b is restricted to me positive, to ensure this the coil current i_c cannot be zero. Hence the state space corresponds to the open subset of $\mathbb{R}^3$ given by:

$$M = (0, \infty) \times \mathbb{R} \times (0, \infty). \tag{3.9}$$

Now, write the dynamical system in an explicitly affine form

$$\underbrace{\begin{pmatrix} \dot{i}_c(t) \\ \dot{v}_b(t) \\ \dot{x}_b(t) \end{pmatrix}}_{\dot{x}(t)} = \underbrace{\begin{pmatrix} \gamma_1 x^1 \\ \gamma_3 (x^1)^2 (x^3)^{-2} + g \\ x^2 \end{pmatrix}}_{v_0(x)} + \underbrace{\begin{pmatrix} \gamma_2 \\ 0 \\ 0 \end{pmatrix}}_{v_1(x)} \underbrace{v_c(t)}_{u(t)} \tag{3.10}$$

Thus the associated distribution $\mathcal{F}$ on the manifold M has fibers $\mathcal{F}_x$ given by:

$$\mathcal{F}_x = \{v_0(x) + \lambda_1 v_1(x) \mid \lambda_1 \in \mathbb{R}\} \tag{3.11}$$

As before check the three hypothesis of the theorem:

1. The distribution $\mathcal{F}$ is strictly affine: Suppose that there exists a $\bar{x}$ such that $v_0(\bar{x}) = 0$

$$\begin{aligned} \gamma_1 \bar{x}^1 &= 0, \\ \gamma_3 (\bar{x}^1)^2 (\bar{x}^3)^{-2} + g &= 0, \\ \bar{x}_2 &= 0 \end{aligned}$$

   It is clear that the first equation implies $\bar{x}_1 = 0$ which is contradictory with the second equation given that $g \neq 0$, hence the distribution is strictly affine.

2. The distribution $\mathcal{F}$ is bracket-generating or almost bracket-generating of constant type: We start with the original distribution $\mathcal{F}$ and from then construct the flag of subsheaves

$$\mathcal{F} = \mathcal{F}^1 \subset \mathcal{F}^2 \subset \cdots \subset TM,$$

   using the definition $\mathcal{F}^{i+1} = \mathcal{F}^i + [\mathcal{F}, \mathcal{F}^i]$.

   (a) $\mathcal{F}^1 = v_1 + \mathrm{span}(v_2)$.

   (b) $\mathcal{F}^2 = \mathcal{F}^1 + [\mathcal{F}, \mathcal{F}^1]$, where

$$[\mathcal{F}, \mathcal{F}^1] = [v_1 + \lambda_2 v_2, v_1 + \beta_2 v_2] = \underbrace{[v_1, v_1]}_{0} + (\beta_2 - \lambda_2) \underbrace{[v_1, v_2]}_{v_3} + \beta_2 \lambda_2 \underbrace{[v_2, v_2]}_{0},$$

   thus the only new vector is $v_3 = [v_1, v_2]$:

$$v_3 = \gamma_1 \gamma_2 \frac{\partial}{\partial x^1} + 2\gamma_3 \gamma_2 x^1 (x^3)^{-2} \frac{\partial}{\partial x^2},$$

   and $\mathcal{F}^2 = v_1 + \mathrm{span}(v_2, v_3)$.

   (c) $\mathcal{F}^3 = \mathcal{F}^2 + [\mathcal{F}, \mathcal{F}^2]$, where

$$\begin{aligned} [\mathcal{F}, \mathcal{F}^2] &= [v_1 + \lambda_2 v_2, v_1 + \beta_2 v_2 + \beta_3 v_3] \\ &= \underbrace{[v_1, v_1]}_{0} + (\beta_2 - \lambda_2) \underbrace{[v_1, v_2]}_{v_3} + \beta_3 \underbrace{[v_1, v_3]}_{v_4} + \beta_2 \lambda_2 \underbrace{[v_2, v_2]}_{0} \lambda_2 \beta_3 \underbrace{[v_2, v_3]}_{v_5}, \end{aligned}$$

   hence the new vectors are $v_4 = [v_1, v_3]$ and $v_5 = [v_2, v_3]$:

$$\begin{aligned} v_4 &= \gamma_1^2 \gamma_2 \frac{\partial}{\partial x^1} + 4x^2 \gamma_2 \gamma_3 x^1 (x^3)^{-3} \frac{\partial}{\partial x^2} + 2\gamma_3 \gamma_2 x^1 (x^3)^{-2} \frac{\partial}{\partial x^3}, \\ v_5 &= -2\gamma_3 \gamma_2^2 (x^3)^{-2} \frac{\partial}{\partial x^2}. \end{aligned}$$

   From the above calculations it follows that the step of the distribution $\mathcal{F}$ is 3, the growth vector is $(1, 2, 3)$ and $\mathcal{F}$ is bracket generating.

3. The distribution $\mathcal{F}$ is homogeneous:

   Start with the frame:

$$
\begin{aligned}
v_1 &= \gamma_1 x^1 \frac{\partial}{\partial x^1} + \left(\gamma_2 \left(\frac{x^1}{x^3}\right)^2 + g\right)\frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3},\\
v_2 &= \gamma_2 \frac{\partial}{\partial x^2},\\
v_3 &= \gamma_1\gamma_2 \frac{\partial}{\partial x^1} + 2\gamma_2\gamma_3 \frac{x^1}{(x^3)^2}\frac{\partial}{\partial x^3},
\end{aligned}
$$

   whose corresponding coframe is:

$$
\begin{aligned}
\eta^1 &= \frac{1}{x^2}\mathrm{d}x^3,\\
\eta^2 &= \frac{1}{\gamma_2}\mathrm{d}x^1 - \frac{\gamma_1(x^3)^2}{2\gamma_2\gamma_3 x^1}\mathrm{d}x^2 + \Big(\frac{\gamma_1 x^1}{2\gamma_2 x^2} + \frac{g\gamma_1(x^3)^2}{2\gamma_2\gamma_3 x^1 x^2} - \frac{\gamma_1 x^1}{\gamma_2 x^2}\Big)\mathrm{d}x^3,\\
\eta^3 &= \frac{(x^3)^2}{2\gamma_2\gamma_3 x^1}\mathrm{d}x^2 - \Big(\frac{x^1}{2\gamma_2 x^2} + \frac{g(x^3)^2}{2\gamma_2\gamma_3 x^1 x^2}\Big)\mathrm{d}x^3.
\end{aligned}
$$

   After taking the derivatives and the wedge products the resulting structure equations are

$$
\begin{aligned}
\mathrm{d}\eta^1 &= T^1_{13}\eta^1 \wedge \eta^3,\\
\mathrm{d}\eta^2 &= T^2_{13}\eta^1 \wedge \eta^3 + T^2_{23}\eta^2 \wedge \eta^3,\\
\mathrm{d}\eta^3 &= T^3_{12}\eta^1 \wedge \eta^3 + T^3_{13}\eta^1 \wedge \eta^3 + T^3_{23}\eta^1 \wedge \eta^3,
\end{aligned}
$$

   where the structure functions are:

$$
\begin{aligned}
T^1_{13} &= \frac{2\gamma_2\gamma_3 x^1}{(x^3)^2 x^2},\\
T^2_{13} &= \gamma_1^2 + \frac{g\gamma_1}{x^2} + \frac{2\gamma_1 x^2}{x^3} - \frac{\gamma_1\gamma_3(x^1)^2}{x^2(x^3)^2},\\
T^2_{23} &= \frac{\gamma_1\gamma_2}{x^1},\\
T^3_{12} &= 1,\\
T^3_{13} &= \frac{2x^2}{x^3} - \frac{\gamma_2(x^1)^2}{x^2(x^3)^2} - \frac{g}{x^2},\\
T^3_{23} &= -\frac{\gamma_2}{x^1}.
\end{aligned}
$$

   It is clear from the structure functions that the distribution is not homogeneous.

Since the final hypothesis is not satisfied the theorem can not be applied and the magnetic levitator dynamical system is not point affine equivalent to any of the normal forms of Theorem 2.1.2.

Appendix A

Transformation of the torsions

This a more direct and long way to calculate the reduction in each step, this is the method used in [1]. It is added for completion but since it is not part of any proof only the first step is calculated:

We have the following coframe transformation: If $(\tilde\theta^1, \tilde\theta^2, \tilde\theta^3)^T$ and $(\theta^1, \theta^2, \theta^3)^T$ are coframes for the principal subbundle B_0 then they are related by the Lie group as

$$\begin{pmatrix}\tilde\theta^1\\ \tilde\theta^2\\ \tilde\theta^3\end{pmatrix} = \begin{pmatrix}1 & 0 & a_3\\ 0 & b_2 & b_3\\ 0 & 0 & c_3\end{pmatrix}^{-1} \begin{pmatrix}\theta^1\\ \theta^2\\ \theta^3\end{pmatrix} \Longleftrightarrow \begin{cases}\theta^1 & = \tilde\theta^1 + a_3\tilde\theta^3,\\ \theta^2 & = b_2\tilde\theta^2 + b_3\tilde\theta^3,\\ \theta^3 & = c_3\tilde\theta^3.\end{cases}$$

On the other hand we have two sets of structure equations:

$$\begin{pmatrix}\mathrm{d}\theta^1\\ \mathrm{d}\theta^2\\ \mathrm{d}\theta^3\end{pmatrix} = \begin{pmatrix}0 & 0 & \alpha_3\\ 0 & \beta_2 & \beta_3\\ 0 & 0 & \gamma_3\end{pmatrix} \wedge \begin{pmatrix}\theta^1\\ \theta^2\\ \theta^3\end{pmatrix} + \begin{pmatrix}T^1_{23} & T^1_{13} & T^1_{12}\\ T^2_{23} & T^2_{13} & T^2_{12}\\ T^3_{23} & T^3_{13} & T^3_{12}\end{pmatrix} \begin{pmatrix}\theta^2\wedge\theta^3\\ \theta^1\wedge\theta^3\\ \theta^1\wedge\theta^2\end{pmatrix}, \tag{A.1}$$

and

$$\begin{pmatrix}\mathrm{d}\tilde\theta^1\\ \mathrm{d}\tilde\theta^2\\ \mathrm{d}\tilde\theta^3\end{pmatrix} = \begin{pmatrix}0 & 0 & \tilde\alpha_3\\ 0 & \tilde\beta_2 & \tilde\beta_3\\ 0 & 0 & \tilde\gamma_3\end{pmatrix} \wedge \begin{pmatrix}\tilde\theta^1\\ \tilde\theta^2\\ \tilde\theta^3\end{pmatrix} + \begin{pmatrix}\tilde T^1_{23} & \tilde T^1_{13} & \tilde T^1_{12}\\ \tilde T^2_{23} & \tilde T^2_{13} & \tilde T^2_{12}\\ \tilde T^3_{23} & \tilde T^3_{13} & \tilde T^3_{12}\end{pmatrix} \begin{pmatrix}\tilde\theta^2\wedge\tilde\theta^3\\ \tilde\theta^1\wedge\tilde\theta^3\\ \tilde\theta^1\wedge\tilde\theta^2\end{pmatrix}. \tag{A.2}$$

Now, $\mathrm{d}\theta^a$ can be expressed in terms of the coframe $\{\tilde\theta^a\}_{a=1,2,3}$ in two forms: first replace θ^a in terms of $\{\tilde\theta^a\}_{a=1,2,3}$ and then calculate the differential or use the corresponding structure equation and replace the $\{\theta^a\}_{a=1,2,3}$ with $\{\tilde\theta^a\}_{a=1,2,3}$.

Take for instance $\mathrm{d}\theta^1$. Replace θ^1 by $\tilde\theta^1 + a_3\tilde\theta^3$ and then operate the differential:

$$\begin{aligned}\mathrm{d}\theta^1 &= \mathrm{d}(\tilde\theta^1 + a_3\tilde\theta^3)\\ &= \mathrm{d}\tilde\theta^1 + \mathrm{d}a_3\wedge\tilde\theta^3 + a_3\mathrm{d}\tilde\theta^3.\end{aligned}$$

Using the structure equations for $\{\tilde\theta^a\}_{a=1,2,3}$

$$\begin{aligned}\mathrm{d}\theta^1 =& (-\tilde\alpha_3 + \mathrm{d}a_3 - a_3\tilde\gamma_3)\wedge\tilde\theta^3 + (\tilde T^1_{23} + a_3\tilde T^3_{23})\tilde\theta^2\wedge\tilde\theta^3 +\\ & (\tilde T^1_{13} + a_3\tilde T^3_{13})\tilde\theta^1\wedge\tilde\theta^3 + (\tilde T^1_{12} + a_3\tilde T^3_{12})\tilde\theta^2\wedge\tilde\theta^3.\end{aligned} \tag{A.3}$$

For the second form first use the structure equations for $\{\theta^a\}$ $a = 1, 2, 3$

$$\mathrm{d}\theta^1 = -\alpha_3 \wedge \eta^3 + T^1_{23}\eta^2 \wedge \eta^3 + T^1_{13}\eta^1 \wedge \eta^3 + T^1_{12}\eta^1 \wedge \eta^2.$$

Now transform the one-forms according to the Lie Group

$$\mathrm{d}\theta^1 = -c_3\alpha \wedge \tilde{\theta}^3 + (T^1_{12}b_2c_3 - T^1_{12}a_3b_2)\tilde{\theta}^2 \wedge \tilde{\theta}^3 + (T^1_{13}c_3 + T^1_{12}b_3)\tilde{\theta}^1 \wedge \tilde{\theta}^3 + T^1_{12}b_2\tilde{\theta}^1 \wedge \tilde{\theta}^2. \quad \text{(A.4)}$$

Hence from equations A.3 and A.4 we have

$$\begin{aligned}(-\tilde{\alpha}_3 + \mathrm{d}a_3 - a_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 + (\tilde{T}^1_{23} + a_3\tilde{T}^3_{23})\tilde{\theta}^2 \wedge \tilde{\theta}^3 + (\tilde{T}^1_{13} + a_3\tilde{T}^3_{13})\tilde{\theta}^1 \wedge \tilde{\theta}^3 + (\tilde{T}^1_{12} + a_3\tilde{T}^3_{12})\tilde{\theta}^1 \wedge \tilde{\theta}^2 \\ = -c_3\alpha \wedge \tilde{\theta}^3 + (T^1_{12}b_2c_3 - T^1_{12}a_3b_2)\tilde{\theta}^2 \wedge \tilde{\theta}^3 + (T^1_{13}c_3 + T^1_{12}b_3)\tilde{\theta}^1 \wedge \tilde{\theta}^3 + T^1_{12}b_2\tilde{\theta}^1 \wedge \tilde{\theta}^2.\end{aligned}$$

Finally we can take the wedge product with $\tilde{\theta}^1$, $\tilde{\theta}^2$ and $\tilde{\theta}^3$ to get

$$\begin{aligned}(-\tilde{\alpha}_3 + \mathrm{d}a_3 - a_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 \wedge \tilde{\eta}^1 + (\tilde{T}^1_{23} + a_3\tilde{T}^3_{23})\tilde{\theta}^2 \wedge \tilde{\theta}^3 \wedge \tilde{\eta}^1 \\ = -c_3\alpha \wedge \tilde{\theta}^3 \wedge \tilde{\eta}^1 + (T^1_{12}b_2c_3 - T^1_{12}a_3b_2)\tilde{\theta}^2 \wedge \tilde{\theta}^3 \wedge \tilde{\eta}^1\end{aligned}$$

$$\begin{aligned}(-\tilde{\alpha}_3 + \mathrm{d}a_3 - a_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 \wedge \tilde{\eta}^2 + (\tilde{T}^1_{13} + a_3\tilde{T}^3_{13})\tilde{\theta}^1 \wedge \tilde{\theta}^3 \wedge \tilde{\eta}^2 \\ = -c_3\alpha \wedge \tilde{\theta}^3 \wedge \tilde{\eta}^2 + (T^1_{13}c_3 + T^1_{12}b_3)\tilde{\theta}^1 \wedge \tilde{\theta}^3 \wedge \tilde{\eta}^2\end{aligned}$$

$$(\tilde{T}^1_{12} + a_3\tilde{T}^3_{12})\tilde{\theta}^1 \wedge \tilde{\theta}^2 \wedge \tilde{\eta}^3 = T^1_{12}b_2\tilde{\theta}^1 \wedge \tilde{\theta}^2 \wedge \tilde{\eta}^3$$

The last equation implies that $T^1_{12}b_2 = \tilde{T}^1_{12} + a_3\tilde{T}^3_{12}$.

Repeating this process with $\mathrm{d}\theta^2$ we obtain:

$$\begin{aligned}(\mathrm{d}b_2 - b_2\tilde{\beta}_2) \wedge \theta^2 + (\mathrm{d}b_3 - b_2\tilde{\beta}_3 - b_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 + (b_2\tilde{T}^2_{23} + b_3\tilde{T}^3_{23})\tilde{\theta}^2 \wedge \tilde{\theta}^3 \\ + (b_2\tilde{T}^2_{13} + b_3\tilde{T}^3_{13})\tilde{\theta}^1 \wedge \tilde{\theta}^3 + (b_2\tilde{T}^2_{12} + b_3\tilde{T}^3_{12})\tilde{\theta}^1 \wedge \tilde{\theta}^2 \\ = -b_2\beta_2 \wedge \tilde{\theta}^2 - (b_3\beta_2 + c_3\gamma_3) \wedge \tilde{\theta}^3 + (T^3_{12}b_2c_3 - T^3_{12}a_3b_2)\tilde{\theta}^2 \wedge \tilde{\theta}^3 \\ + (T^3_{13}c_3 + T^3_{12}b_3)\tilde{\theta}^1 \wedge \tilde{\theta}^3 + T^3_{12}b_2\tilde{\theta}^1 \wedge \tilde{\theta}^2. \qquad \text{(A.5)}\end{aligned}$$

Now taking the wedge product with $\tilde{\theta}^1$, $\tilde{\theta}^2$ and $\tilde{\theta}^3$:

$$\begin{aligned}(\mathrm{d}b_2 - b_2\tilde{\beta}_2) \wedge \theta^2 \wedge \tilde{\theta}^1 + (\mathrm{d}b_3 - b_2\tilde{\beta}_3 - b_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^1 + (b_2\tilde{T}^2_{23} + b_3\tilde{T}^3_{23})\tilde{\theta}^2 \wedge \tilde{\theta}^3 \\ = -b_2\beta_2 \wedge \tilde{\theta}^2 \wedge \tilde{\theta}^1 + (b_3\beta_2 + c_3\gamma_3) \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^1 + (T^3_{12}b_2c_3 - T^3_{12}a_3b_2)\tilde{\theta}^2 \wedge \tilde{\theta}^3,\end{aligned}$$

$$\begin{aligned}(\mathrm{d}b_3 - b_2\tilde{\beta}_3 - b_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^2 + (b_2\tilde{T}^2_{13} + b_3\tilde{T}^3_{13})\tilde{\theta}^1 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^2 \\ = (b_3\beta_2 + c_3\gamma_3) \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^2 + (T^3_{13}c_3 + T^3_{12}b_3)\tilde{\theta}^1 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^2,\end{aligned}$$

$$(\mathrm{d}b_2 - b_2\tilde{\beta}_2) \wedge \theta^2 \wedge \tilde{\theta}^3 + (b_2\tilde{T}^2_{12} + b_3\tilde{T}^3_{12})\tilde{\theta}^1 \wedge \tilde{\theta}^2 = -b_2\beta_2 \wedge \tilde{\theta}^2 \wedge \tilde{\theta}^3 + T^3_{12}b_2\tilde{\theta}^1 \wedge \tilde{\theta}^2.$$

In this instance no information can be obtained from these equations.

Now for $\mathrm{d}\theta^3$

$$\begin{aligned}(\mathrm{d}c_3 - c_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 + c_3\tilde{T}^3_{23}\tilde{\theta}^2 \wedge \tilde{\theta}^3 + &\\ c_3\tilde{T}^3_{13}\tilde{\theta}^1 \wedge \tilde{\theta}^3 + c_3\tilde{T}^2_{12}\tilde{\theta}^1 \wedge \tilde{\theta}^2 &\\ = -c_3\gamma_3 \wedge \tilde{\theta}^3 + (T^3_{12}b_2c_3 - T^3_{12}a_3b_2)\tilde{\theta}^2 \wedge \tilde{\theta}^3 + &\\ (T^3_{13}c_3 + T^3_{12}b_3)\tilde{\theta}^1 \wedge \tilde{\theta}^3 + T^3_{12}b_2\tilde{\theta}^1 \wedge \tilde{\theta}^2.&\end{aligned}$$

$$\begin{aligned}(\mathrm{d}c_3 - c_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^1 c_3\tilde{T}^3_{23}\tilde{\theta}^2 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^1 &= -c_3\gamma_3 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^1 (T^3_{12}b_2c_3 - T^3_{12}a_3b_2)\tilde{\theta}^2 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^1,\\ (\mathrm{d}c_3 - c_3\tilde{\gamma}_3) \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^2 c_3\tilde{T}^3_{13}\tilde{\theta}^1 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^2 &= -c_3\gamma_3 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^2 (T^3_{13}c_3 + T^3_{12}b_3)\tilde{\theta}^1 \wedge \tilde{\theta}^3 \wedge \tilde{\theta}^2,\\ c_3\tilde{T}^2_{12}\tilde{\theta}^1 \wedge \tilde{\theta}^2 \wedge \tilde{\theta}^3 &= T^3_{12}b_2\tilde{\theta}^1 \wedge \tilde{\theta}^2 \wedge \tilde{\theta}^3.\end{aligned}$$

The last equation implies $c_3\tilde{T}^2_{12} = T^3_{12}b_2$. Hence we have two equations

$$c_3\tilde{T}^2_{12} = T^3_{12}b_2, \qquad\qquad T^1_{12}b_2 = \tilde{T}^1_{12} + a_3\tilde{T}^3_{12}.$$

Which can be written as

$$\tilde{T}^1_{12} = b_2T^1_{12} - \frac{a_3b_2}{c_3}T^3_{12}, \qquad\qquad \tilde{T}^3_{12} = \frac{b_2}{c_3}T^3_{12}. \tag{A.6}$$

These equations express the transformation of the torsion functions due to the group action on the coframe (from the coframe $\{\theta^a\}_{a=1,2,3}$ to the coframe $\{\tilde{\theta}^a\}_{a=1,2,3}$).

Some torsion functions do not appear in these equations, this is because the action of the group does not transform them into other torsion functions and as such cannot be reduced yet.

Equations (A.6) imply that we can take any coframe in $\{\theta^a\}_{a=1,2,3}$ and using the group action transform it into a coframe where $T^1_{12} = 0$ and $T^3_{12} = 1$. Thus we can use these particular torsion functions to continue the analysis. In order to guarantee this condition we need to make sure of to preserve these torsion functions, which can only be done making $a_3 = 0$ and $b_2 = c_3$. This is the Cartan reduction step, and in the following steps we can fix $T^1_{12} = 0$ and $T^3_{12} = 1$ as long as $a_3 = 0$ and $b_2 = c_3$.

This changes the structure equations and then the process repeats.

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