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Eigenvalues Of A Hessenberg-Toeplitz Matrix

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Chapter 1

Introduction

The $n \times n$ Toeplitz matrix generated by a complex-valued function $a \in L^1(\mathbb{T})$, on the complex unit circle $\mathbb{T}$, is the square matrix

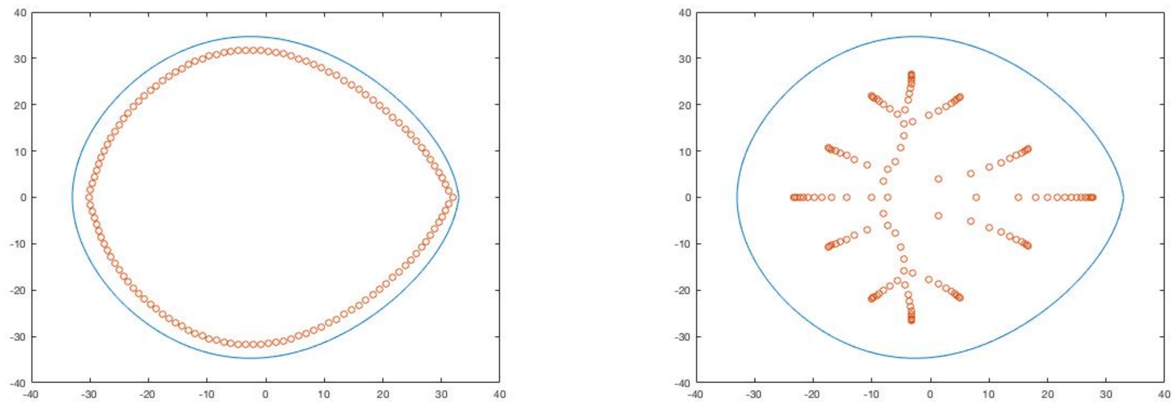
$$T_n(a) = (a_{j-k})_{j,k=0}^{n-1},$$

where a_k is the kth Fourier coefficient of a, that is,

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(1+\mathrm{e}^{i\theta}) + \mathrm{e}^{-ik\theta}\,\mathrm{d}\theta = \frac{1}{2\pi i}\int_{\mathbb{T}} a(t)t^{-(k+1)}\,\mathrm{d}t \quad (k \in \mathbb{Z}).$$

The function a is referred to as the symbol of the matrices $T_n(a)$.

Denote by H^∞ the usual Hardy space of (boundary values of) bounded analytic functions over the unit disk $\mathbb{D}$. For a function $a \in C(\mathbb{T})$, let $\mathrm{wind}_\lambda(a)$ be the winding number of a about the point $\lambda \in \mathbb{C} \setminus \mathcal{R}(a)$ where $\mathcal{R}(a)$ stands for the range of a, and let $\mathcal{D}(a)$ be the set $\{\lambda \in \mathbb{C} \setminus \mathcal{R}(a)\colon\ \mathrm{wind}_a(\lambda) \neq 0\}$. Let $\operatorname{sp} T_n(a)$ be the spectrum of a Toeplitz matrix by be the set $\{\lambda\colon \mathcal{D}_n(a-\lambda) = 0\}$ where $\mathcal{D}_n(a-\lambda)$ is determinant of $T_n(a-\lambda)$, we say that spectrum may have canonical or skin distribution if $d_H(R(a), \operatorname{sp} T_n(a)) \to 0$ when $n \to 0$ (see Figure 1.1a) and Skeleton distribution if $d_H(R(a), \operatorname{sp} T_n(a)) \nrightarrow 0$ when $n \to 0$ (see Figure 1.1b) where d_H is the Haussdorff distance.



(a) Skin distribution

(b) Skeleton distribution

Figure 1.1: The pictures shows the range $\mathcal{R}(a)$ (blue color) for the symbol $a(t) = \frac{1}{t}(33 - (t + t^2)(1 + t^2)^{\frac{3}{4}})$, $\operatorname{sp} T_{128}(a)$ in Figure 1.1a and $\operatorname{sp} T_{512}$ in Figure 1.1b (orange color). The spectrum was calculated using *MATLAB*.

For a real-valued symbol a, the matrices $T_n(a)$ are all Hermitian, and in this case a number of results on the asymptotics of the eigenvalues of $T_n(a)$ are known; see, for example, [4, 5, 8–13, 15–18, 20]. If a is a rational function, in [6, 7, 14] describe the limiting behavior of the eigenvalues of $T_n(a)$. If a is a non-smooth symbol, in [19, 21] are devoted to the asymptotic eigenvalue distribution. If $a \in L^\infty(\mathbb{T})$ and $\mathcal{R}(a)$ does not separate the plane, in [19, 24] it is prove that the eigenvalues of $T_n(a)$ approximate $\mathcal{R}(a)$. Many of the results of the in cited above can also be found in [23, 25, 29].

In 1990, Widom [19] showed that if $\mathcal{R}(a)$ is a Jordan curve and a is smooth on $\mathbb{T}$ minus a single point but not smooth on all of $\mathbb{T}$, then the spectrum of $T_n(a)$ has canonical distribution. He also raised the following intriguing conjecture, which is still an open problem:

The eigenvalues of $T_n(a)$ are canonically distributed except when
a extends analytically to an annulus $r < |z| < 1$ or $1 < |z| < R$.

Reference [3] deals with asymptotic formulas for individual eigenvalues of Toeplitz matrices whose symbols are complex-valued and have a so-called Fisher–Hartwig singularity. These are special symbols that are smooth on $\mathbb{T}$ minus a single point but not smooth on the entire circle $\mathbb{T}$; see [23, 25].

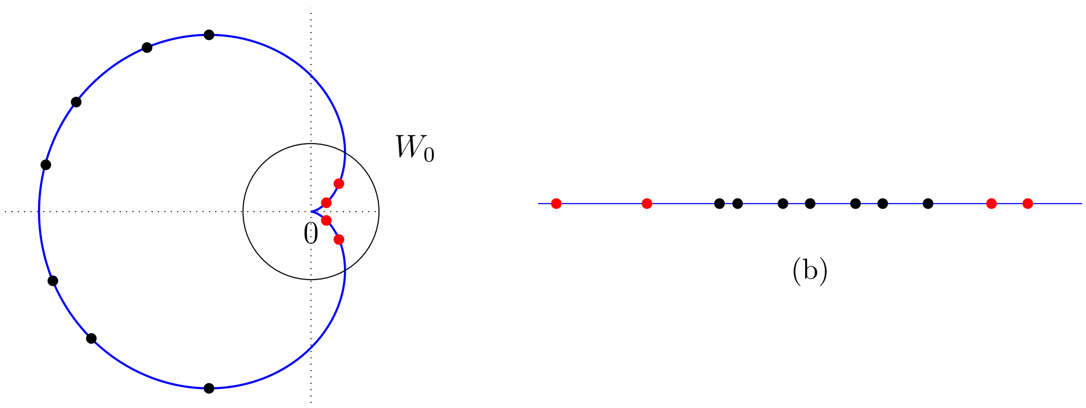
We consider here genuinely complex-valued symbols, in which case less is known. Dai, Geary, and Kadanoff [3] considered symbols of the form

$$a(t) = \left(2 - t - \frac{1}{t}\right)^{\gamma}(-t)^{\beta} = \frac{(-1)^{\beta+3\gamma}}{t^{\gamma-\beta}}(1-t)^{2\gamma} \quad (t \in \mathbb{T}),$$

where $0 < \gamma < -\beta < 1$. They conjectured that the eigenvalues $\lambda = \lambda_j^{(n)}$ satisfy

$$\lambda_j^{(n)} \sim a\left(n^{\frac{1}{n}(2\gamma-1)} \mathrm{e}^{-\frac{1}{n}2\pi i j}\right) \quad (j = 0, \dots, n-1), \tag{1.1}$$

and confirmed this conjecture numerically. Note that in (1.1) the argument of a can be outside of $\mathbb{T}$. This is no problem, since a can be extended analytically to a neighborhood of $\mathbb{T} \setminus \{1\}$ not containing the singular point 1.



(a)

(b)

Figure 1.2: Figura a Es la figura 1 y figura b es la otra

In the following work we will study eigenvalues of $T_n(a)$ for symbols of the form

$$a(t) = \frac{1}{t}(1-t)^{\alpha} f(t) \quad (t \in \mathbb{T}), \tag{1.2}$$

where $0 < \alpha < 1$, which we will divide into two parts. Let W_0 be any small open neighborhood of the origin in $\mathbb{C}$. In the first part we will study the eigenvalues outside of W_0, those will be called **inner eigenvalues** (red points in Figure 1.2a) and in the second part the eigenvalues inside of W_0, those will be called **extreme eigenvalues** (black points in Figure 1.2a). Those names are usual in Toeplitz operators literature, however the natural explanation of those names is that if we cut the cardioid by the origin point (see Figure 1.2a), stretched it to a line segment (see Figure 1.2b), the black points are located in the inner part of the segment, and the red points are located in the extremes of the segment.

According to [19], in our case the spectrum of $T_n(a)$ has canonical distribution, that is the Haussdorff distance between the spectrum of $T_n(a)$ and $\mathcal{R}(a)$ goes to zero when n goes to infinity. Note that when $\beta = \gamma - 1$ and $f \equiv 1$ our symbol coincides with the one of [3].

This work consists of studying and complementing the papers [1], [2]. In Chapter 2 we give the preliminaries for understanding the following Chapters. In Chapter 3 we state the main results for each case of the eigenvalues (inner and extreme) giving a sketch of how to solve these and present the necessary tools to prove them. In Chapter 4 we give a key example when the symbol a equals $\frac{1}{t}(1-t)^{\frac{3}{4}}$ using the main results given on Chapter 3.

In Chapter 5 we study the behavior of inner eigenvalues, prove on detail the main results and show that the conjecture (1.1) in the special case $\beta = \gamma - 1$ is true for the inner eigenvalues. We will also give an asymptotic approximation for each individual eigenvalue incorporating two terms.

Similarly in Chapter 6 we study the behavior of extreme eigenvalues, prove on detail the main results and show that the problem to find the extreme individual eigenvalues of $T_n(a)$, as n goes to infinity, can be reduced to the solution of a certain equation in a fixed complex domain not depending on n. In this sense our results extend to the complex-value case the well known results of Parter [12] and Widom [19] for the real-value case. Moreover, we show that the conjecture (1.1) of Dai, Geary, and Kadanoff [3] is not true for the extreme eigenvalues.

In conclusion, in the Chapters 5 and 6 we obtain an univocal correspondence between eigenvalues and some elements of the domain corresponding to the extension of the symbol a, also for the inner eigenvalues there is a relationship with the nth root of unity (eigenvalues enumerable and uniformly distanced), and for the extreme eigenvalues there is a relationship with the zeros of an analytic function, thus is only necessary to find those zeros once.

For the eigenvalues of $T_n(a)$ regardless the case (inner or extreme), we will give an asymptotic approximation depending only on n and its respective relationships, therefore we can approximate the eigenvectors. It is important to note that no matter the values of n, because for example, in the conjecture (1.1) of Dai, Gearay and Kadanoff [3], the interested n is approximately the Avogadro number.

Chapter 2

Preliminary

In this chapter we mention some notions will be needing in this work.

Definition 2.1 (Hessenberg Matrix)**.** Let A be an square $n \times n$ matrix.

- Upper Hessenberg matrix: A is said to be in upper Hessenberg form or to be an upper Hessenberg matrix if $a_{i,j} = 0$ for all i, j with $i > j + 1$.

- Lower Hessenberg matrix: A is said to be in lower Hessenberg form or to be an lower Hessenberg matrix if its transpose is an upper Hessenberg matrix, or equivalently, if $a_{i,j} = 0$ for all i, j with $j > i + 1$.

In this work, when we mention the Hessenberg matrix, we mean a lower Hessenberg matrix.

Definition 2.2 (Hausdorff Distance)**.** Let X and Y be two non-empty subsets of a metric space (M, d). We define their Hausdorff distance $d_H(X, Y)$ by

$$d_{\mathrm{H}}(X,Y) = \max\{\sup_{x\in X} \inf_{y\in Y} d(x,y), \sup_{y\in Y} \inf_{x\in X} d(x,y)\}$$

where sup represents the supremum and inf the infimum.

Definition 2.3 (Hardy Space)**.** The Hardy spaces (or Hardy classes) H^p are certain spaces of holomorphic functions on the unit disk or upper half plane satisfying

$$\sup_{0\leqslant r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,\mathrm{d}\theta\right)^{\frac{1}{p}}<\infty.$$

This class H^p is a vector space. The number on the left side of the above inequality is the Hardy space p-norm for f, denoted by $\|f\|_{H^p}$. It is a norm when $p \geqslant 1$, but not when $0 < p < 1$.

The space H^∞ is defined as the vector space of bounded holomorphic functions on the disk, with the norm

$$\|f\|_{H^\infty} = \sup_{|z|<1}|f(z)|.$$

Theorem 2.4 (Lebesgue's Dominated Convergence Theorem)**.** *Let $\{f_n\}_{n=1}^\infty$ be a sequence of complex-valued measurable functions on a measure space (S, Σ, μ). Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that $|f_n(x)| \leqslant g(x)$ for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable and*

$$\lim_{n\to\infty}\int_S |f_n - f|\,\mathrm{d}\mu = 0$$

which also implies

$$\lim_{n\to\infty}\int_S f_n\,\mathrm{d}\mu = \int_S f\,\mathrm{d}\mu.$$

2.1 Toeplitz matrix

Let $a \in L^1(\mathbb{T})$ be a symbol defined as in the Introduction with

$$a(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2.1}$$

where $a_0 \neq 0$, and consider $c(z) = \dfrac{1}{a(z)} = \displaystyle\sum_{m=0}^{\infty} c_m z^n$ with $c_0 = a_0^{-1}$.

Proposition 2.5 (Baxter-Schmidt Formula for Toeplitz determinants)**.** *If n, $r \geqslant 1$, then*

$$a_0^{-r} D_n(z^{-r}a) = (-1)^{rn} c_0^{-n} D_r(z^{-n}c).$$

Proposition 2.6. *Let a be a symbol defined as* (2.1) *and $b \in L^1(\mathbb{T})$ then*

$$T_n(ab) = T_n(a)T_n(b).$$

The inverse of a Toeplitz matrix is not always a Toeplitz matrix, but the previous proposition shows that if $a(z) = \sum_{n=0}^{\infty} a_n z^n$ then $T_n^{-1}(a) = T_n(c)$.

Let P_n be the projection in $\ell^2(\mathbb{C})$ defined by

$$P_n \colon \{z_0, z_1, z_3, ...\} \longmapsto \{z_0, z_2, \cdots, z_{n-1}, 0, ...\}.$$

Proposition 2.7 (Finite section method)**.** *Let X be $\ell^2(\mathbb{C})$ and a be a symbol defined as* (2.1)*. If $T(a)$ is invertible, then the operators $T_n^{-1}(a)P_n$ converge strongly to $T^{-1}(a)$ in X*

$$\textit{i.e.} \quad \|T_n^{-1}(a)P_n x - T^{-1}(a)x\| \longrightarrow 0 \quad \textit{for all} \quad x \in X,$$

where $T(a)$ is an infinite Toeplitz matrix.

By functional analysis theory [30], the reader can verify that ℓ^2 is isomorphic to L^2, where the isomorphism is given by the Fourier Transform. The Propositions 2.5, 2.6 and 2.7 are classic and known results in the literature of the Toeplitz matrices, the proof can found in [29] and [22].

2.2 Asymptotic analysis

Definition 2.8. Let f and $\phi \colon D \subseteq \mathbb{C} \to \mathbb{C}$ be functions. We say that:

- $f = O(\phi)$ when $(z \to z_0)$, if $\forall U(z_0) \subseteq \mathbb{C}\ \exists A > 0$ such that $|f(z)| \leqslant A|\phi(z)|$ for $z \in U$.

- $f = o(\phi)$ when $(z \to z_0)$, if $\forall \varepsilon > 0\ \exists U_\varepsilon(z_0) \subseteq \mathbb{C}$ such that $|f(z)| \leqslant \varepsilon|\phi(z)|$ for $z \in U_\varepsilon$.

- $f \sim \phi$ when $(z \to z_0)$, if $f(z) = \phi(z)(1 + o(1))$.

Proposition 2.9 (Properties of o and O)**.** *Let f and $\phi \colon D \subseteq \mathbb{C} \to \mathbb{C}$ be functions.*

- *For k constant we have $kO(f) = O(f)$, similarly with o.*

- *$o(f) = O(f)$ but $O(f) = o(f)$ is not always true.*

- *$O(f) + o(f) = O(f)$.*

- *$f = O(1)$ in D if only if $|f|$ is bounded in D.*

- *$f = o(1)$ in $(z \to z_0)$ if only if $\lim\limits_{z \to z_o} f(z) = 0$.*

Note that "=" here is not usual equal, since, for example $o(1) = O(1)$ but $O(1) \neq o(1)$ because when $x \to \infty$, we have $\sin(x) = O(1)$ but $\sin(x) \neq o(1)$, however $\mathrm{e}^{-x} = o(1)$ and also $\mathrm{e}^{-x} = O(1)$.

Definition 2.10 (Piecewise)**.** A piecewise function $C_{\mathrm{pw}}[a, b]$ is a continue function almost everywhere.

Proposition 2.11 (Riemann Lebesgue's lemma)**.** *Let $q \in C_{pw}[a, b]$ then*

$$Q(x) = \int_a^b \mathrm{e}^{ixt} q(t)\, \mathrm{d}t = o(1), \quad (x \to \infty).$$

Theorem 2.12. *Let $\beta > 0$, $\delta > 0$, $v \in C^\infty[0, \delta]$, $v^{(s)}(\delta) = 0$ for all $s \geqslant 0$. Then, as $n \to \infty$,*

$$\int_0^\delta \theta^{\beta-1} v(\theta) \mathrm{e}^{in\theta}\, \mathrm{d}\theta \sim \sum_{s=0}^{\infty} \frac{a_s}{n^{s+\beta}},$$

where

$$a_s = \frac{v^{(s)}(0)}{s!} \Gamma(s + \beta) i^{s+\beta} \tag{2.2}$$

and $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1} \mathrm{e}^{-t} dt$ is Euler's Gamma function.

Proof. Since $v \in C^\infty[0,\delta]$ we can replace $v(\theta)$ with its Taylor's series centered on 0 then

$$\int_0^\delta \theta^{\beta-1} \sum_{s=0}^{\infty} \frac{v^{(s)}(0)\theta^{(s)}}{s!} \mathrm{e}^{in\theta}\,\mathrm{d}\theta = \sum_{s=0}^{\infty} \frac{v^{(s)}(0)}{s!} \int_0^\delta \theta^{s+\beta-1}\mathrm{e}^{in\theta}\,\mathrm{d}\theta.$$

This equality is true by the dominated convergence Theorem 2.4. Now

$$\int_0^\delta \theta^{s+\beta-1}\mathrm{e}^{in\theta}\,\mathrm{d}\theta = \int_0^\infty \theta^{s+\beta-1}\mathrm{e}^{in\theta}\,\mathrm{d}\theta - \int_\delta^\infty \theta^{s+\beta-1}\mathrm{e}^{in\theta}\,\mathrm{d}\theta.$$

Using the Riemann Lebesgue's lemma (2.11) when $n \to \infty$ we have $\int_\delta^b \theta^{s+\beta-1}\mathrm{e}^{in\theta}\,\mathrm{d}\theta = o(1)$ then $\int_\delta^\infty \theta^{s+\beta-1}\mathrm{e}^{in\theta}\,\mathrm{d}\theta = o(1)$. Taking the variable change $-\tau = in\theta$, we get

$$\int_0^\infty \theta^{s+\beta-1}\mathrm{e}^{in\theta}\,\mathrm{d}\theta = \left(\frac{-1}{in}\right)^{\beta+s}, \qquad \int_0^\infty \tau^{s+\beta-1}\mathrm{e}^{-\tau}d\tau = \left(\frac{i}{n}\right)^{\beta+s}\Gamma(s+\beta),$$

thus

$$\int_0^\delta \theta^{\beta-1} \sum_{s=0}^{\infty} \frac{v^{(s)}(0)\theta^{(s)}}{s!} \mathrm{e}^{in\theta}\,\mathrm{d}\theta = \sum_{s=0}^{\infty} \frac{1}{n^{\beta+s}} \frac{v^{(s)}(0)}{s!}\Gamma(s+\beta)i^{\beta+s}.$$

$\square$

Corollary 2.13. *Using the same hypothesis of theorem 2.12, we get*

$$\int_0^\delta \theta^{\beta-1} v(\theta)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta \sim \sum_{s=0}^{\infty} \frac{\overline{a}_s}{n^{s+\beta}}.$$

For more properties of the order symbol, or the proof of Proposition 2.11, we refer the reader to [28].

2.3 Complex analysis

Proposition 2.14. *Let f be an analytic function at z_0 and $f(z) = f(z_0) + O(|z - z_0|)$ then*

$$f(z_0 + \Delta z) = f(z_0) + O(|\Delta z|) \quad (\Delta z \to 0).$$

Theorem 2.15 (Rouché). *For any two complex-valued functions f and g holomorphic inside some region K with closed and simple contour ∂K, if $|g(z)| < |f(z)|$ on ∂K, then f and $f+g$ have the same number of zeros inside K, where each zero is counted as many times as its multiplicity.*

Theorem 2.16 (Maximum modulus Principle). *Let f be a complex holomorphic function on some connected open subset D of $\mathbb{C}$. If z_0 is a point in D such that*

$$|f(z_0)| \geqslant |f(z)|$$

for all z in a neighborhood of z_0, then the function f is constant on D.

Corollary 2.17 (Maximum Modulus Principle). *Let D be an open subset and bounded in $\mathbb{C}$, $f\colon \overline{D} \to D$ be continuous in $\overline{D}$ and holomorphic in D, then*

$$\sup\{|f(z)| : z \in \overline{D}\} = \sup\{|f(z)| : z \in \partial D\}.$$

Chapter 3

Main results

In this chapter we propose the necessary conditions that must be satisfied by the symbol a to be able to find the inner and extreme eigenvalues (i.e solve $D_n(a-\lambda)=0$). For each case the conditions may be different because of the nature of the problem since in the particular case of extreme eigenvalues we have the condition of non-differentiability at 1. Also we will give a sketch of how to solve each case.

3.1 Inner eigenvalues

Properties 3.1. Let $a(t)=\dfrac{1}{t}h(t)$ be a symbol, where $a\in C(\mathbb{T})$ then:

1. $h(t)=(1-t)^{\alpha}f(t)$, where $\alpha\in[0,\infty)\setminus\mathbb{Z}$ and $f\in C^{\infty}(\mathbb{T})$;

2. $h\in H^{\infty}$ and $h_0\neq 0$;

3. h has an analytic extension to an open neighborhood W of $\mathbb{T}\setminus\{1\}$ not containing the point 1;

4. $\mathcal{R}(a)$ is a Jordan curve in $\mathbb{C}$, $\operatorname{wind}_{\lambda}(a)=-1$ for each $\lambda\in\mathcal{D}(a)$, and $a'(t)\neq 0$ for every $t\in\mathbb{T}\setminus\{1\}$.

Here h_0 is the zeroth Fourier coefficient of h.

For the farthest from zero eigenvalues, we study the eigenvalues of $T_n(a)$ for symbols a that satisfy the Properties 3.1.

Let $D_n(a)$ denote the determinant of $T_n(a)$. Thus, the eigenvalues λ of $T_n(a)$ are the solutions of the equation $D_n(a-\lambda) = 0$. The assumptions imply that $T_n(a)$ is a Hessenberg matrix. This circumstance together with the Baxter-Schmidt Formula 2.5 for Toeplitz determinants allows us to express $D_n(a-\lambda)$ as a Fourier integral. The value of this integral mainly depends on λ and a power of the singularity of $(1-t)^\alpha$ at the point 1. Let W_0 be a small open neighborhood of zero in $\mathbb{C}$. We show that for every point $\lambda \in \mathcal{D}(a) \cap \big(a(W) \setminus W_0\big)$ there exists a unique point $t_\lambda \notin \overline{\mathbb{D}}$ such that $a(t_\lambda) = \lambda$. After exploring the contributions of λ and the singular point 1 to the Fourier integral, we get the following asymptotic expansion for $D_n(a-\lambda)$.

Theorem 3.1 (Refer [2]). *Let a be the symbol satisfying the Properties 3.1. Then, for every small open neighborhood W_0 of zero in $\mathbb{C}$ and every $\lambda \in \mathcal{D}(a) \cap \big(a(W) \setminus W_0\big)$,*

$$D_n(a-\lambda) = (-h_0)^{n+1}\left(\frac{1}{t_\lambda^{n+2}a'(t_\lambda)} - \frac{f(1)\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi\lambda^2 n^{\alpha+1}} + R_1(n,\lambda)\right), \tag{3.1}$$

where $R_1(n,\lambda) = O\Big(\frac{1}{n^{\alpha+\alpha_0+1}}\Big)$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$ and here $\alpha_0 = \min\{\alpha, 1\}$.

The first term in brackets is the contribution of λ, while the second is the contribution of the point 1.

Now, here are our main results. Put $\omega_n := \exp\left(\frac{-2\pi i}{n}\right)$, for each n there exist integers n_1 and n_2 such that $\omega_n^{n_1}, \omega_n^{n-n_2} \in a^{-1}(W_0)$ but $\omega_n^{n_1+1}, \omega_n^{n-n_2-1} \notin a^{-1}(W_0)$. Recall that $a(t_\lambda) = \lambda$.

Theorem 3.2 (Refer [2]). *Let a be the symbol satisfying the Properties 3.1. Then, for every small open neighborhood W_0 of the origin in $\mathbb{C}$ and every j between n_1 and $n-n_2$,*

$$t_{\lambda_{j,n}} = n^{\frac{\alpha+1}{n}}\omega_n^j\left(1 + \frac{1}{n}\log\left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}}\right) + R_2(n,j)\right), \tag{3.2}$$

where $R_2(n,j) = O\Big(\frac{1}{n^{\alpha_0+1}}\Big) + O\Big(\frac{\log n}{n^2}\Big)$ *as* $n \to \infty$*, uniformly with respect to* j *in* $(n_1, n-n_2)$*. Here* $\alpha_0 = \min\{\alpha, 1\}$ *and*

$$C_1 = \frac{f(1)\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi}.$$

Formula (3.2) proves conjecture (1.1) in the special case $\beta = \gamma - 1$. It shows that as n increases, the point $t_{\lambda_{j,n}}$ is close to $n^{\frac{\alpha+1}{n}}\omega_n^j$. Finally, we apply a at both sides of (3.2) to obtain the following expression for $\lambda_{j,n}$.

Theorem 3.3 (Refer [2])**.** *Let a be the symbol satisfying the Properties 3.1. Then, for every small neighborhood W_0 of zero in $\mathbb{C}$ and every j between n_1 and $n-n_2$,*

$$\lambda_{j,n} = a(\omega_n^j) + (\alpha+1)\,\omega_n^j a'(\omega_n^j)\frac{\log n}{n} + \frac{\omega_n^j a'(\omega_n^j)}{n}\log\left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}}\right) + R_3(n,j), \quad (3.3)$$

where C_1 is as in Theorem 3.2 and $R_3(n,j) = O\Big(\frac{1}{n^{\alpha_0+1}}\Big) + O\Big(\frac{\log n}{n^2}\Big)$ *as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$.*

We remark that we wrote down only the first few terms in the asymptotic expansions but that our method is constructive and would allow us to get as many terms as we desire. Clearly, conjecture (1.1) corresponds to the first term in the asymptotic expansion (3.2).

Figure 3.1 illustrates Theorem 3.3. We present another simulation graphic and error tables made with *MATLAB* software to show that incorporating the second term of the expansion (3.2) (= third term in (3.3)) reduces the error to nearly one tenth.

3.2 Extreme eigenvalues

We take the multi-valued complex function $z \mapsto z^\beta$ ($\beta \in \mathbb{R}$) with the branch specified by $-\pi < \arg z^\beta \leqslant \pi$. Let $B(z_0, r)$ be the set $\{z \in \mathbb{C} \colon |z - z_0| < r\}$.

Properties 3.2. Let symbols $a(t) = \dfrac{1}{t}(1-t)^\alpha f(t)$ where $a \in C(\mathbb{T})$ then:

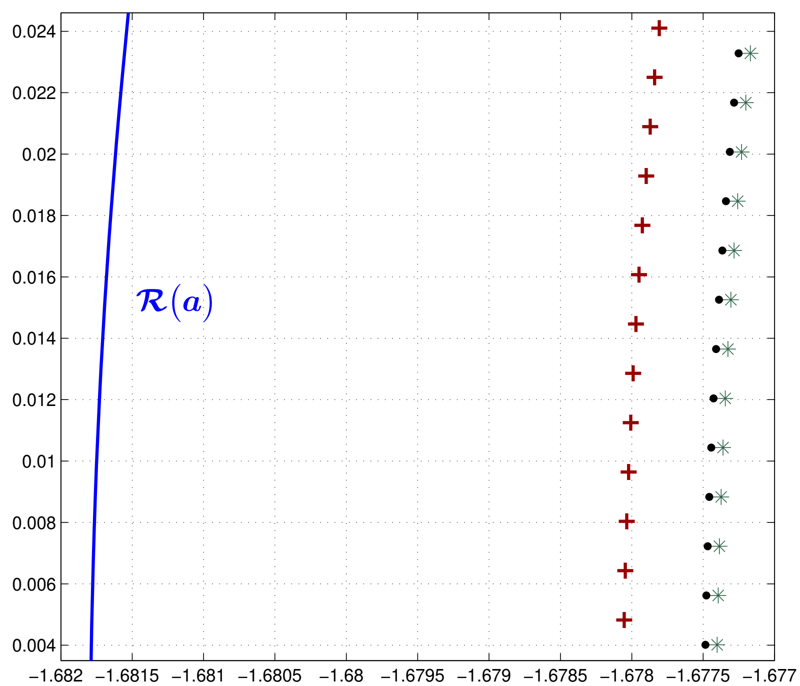


Figure 3.1: (Refer [2]) The picture shows a piece of $\mathcal{R}(a)$ for the symbol $a(t) = \frac{1}{t}(1-t)^{\frac{3}{4}}$ (solid line) located "far" from zero. The dots are $\operatorname{sp} T_{4096}(a)$ calculated by *MATLAB*. The crosses and the stars are the approximations obtained by using 2 and 3 terms of (3.3), respectively.

1. The function f is in H^∞ with $f(0) \neq 0$ and for some $\varepsilon > 0$, f has an analytic continuation to the region

$$K_\varepsilon \coloneqq B(1,\varepsilon) \setminus \{x \in \mathbb{R} \colon 1 < x < 1+\varepsilon\}$$

and is continuous in $\hat{K}_\varepsilon \coloneqq \overline{B(1,\varepsilon)} \setminus \{x \in \mathbb{R} \colon 1 < x \leqslant 1+\varepsilon\}$. Additionally, $f_\varphi(x) \coloneqq f(1+x+\mathrm{e}^{i\varphi})$ belongs to the algebra $C^2[0,\varepsilon)$ for each $-\pi < \varphi \leqslant \pi$.

2. Let $0 < \alpha < 1$ be a constant and take

$$-\alpha\pi < \arg(1-z)^\alpha \leqslant \alpha\pi \text{ when } -\pi < \arg(1-z) \leqslant \pi.$$

3. $\mathcal{R}(a)$ is a Jordan curve in $\mathbb{C}$ and $\operatorname{wind}_\lambda(a) = -1$ for each $\lambda \in \mathcal{D}(a)$.

For the extreme eigenvalues, we study the eigenvalues of $T_n(a)$ for symbols a that satisfy the Properties 3.2.

Note that, in general, $\lim_{\varphi\to 0^+} f_\varphi(x) \neq \lim_{\varphi\to 0^-} f_\varphi(x)$, thus f cannot be continuously extended to the ball $B(1,\varepsilon)$. Without loss of generality, we assume that $f(1) = 1$.

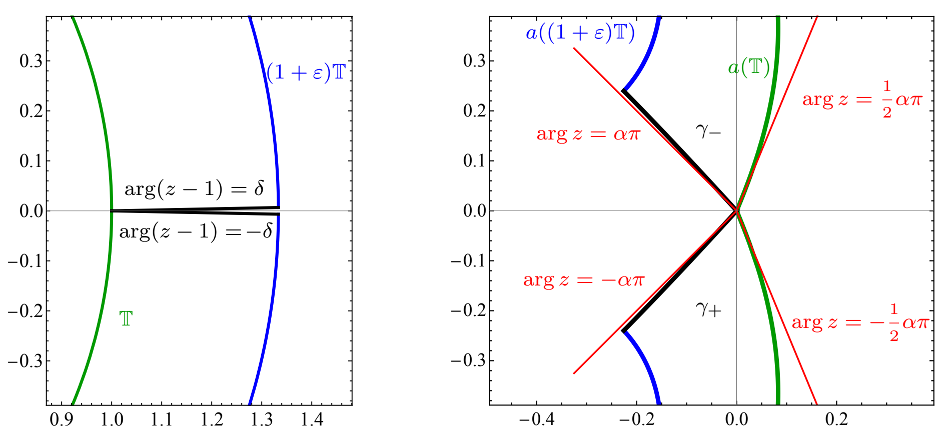


Figure 3.2: (Refer [1]) The behavior of the function $a(t) = \frac{1}{t}(1-t)^{\frac{3}{4}}$ near the point $t = 1$.

Before we go further, we need to give the required understanding to the symbol a near to the point $t = 1$. For $0 \leqslant x \leqslant \varepsilon$ take

$$a_+(x) \coloneqq \lim_{\delta \to 0^+} a(1 + x\mathrm{e}^{i\delta}), \qquad a_-(x) \coloneqq \lim_{\delta \to 0^-} a(1 + x\mathrm{e}^{i\delta}),$$

$$\gamma_+ \coloneqq a_+([0, \varepsilon]), \qquad \gamma_- \coloneqq a_-([0, \varepsilon]).$$

The Figure 3.2 shows the situation. Note that γ_+ and γ_- are very close to the lines $\arg z = \mp\alpha\pi$, respectively, but they are not the same. In Chapter (5) we will prove that if $\lambda \in \mathcal{D}(a)$ is bounded away from 0, then a can be extended bijectively to a certain neighborhood of $\mathbb{T} \setminus \{1\}$ not containing the point 1, but if λ is arbitrarily close to 0 the situation is much more complicated. The map $z \mapsto z^\alpha$ transforms the real negative semi-axis into the lines $\arg z = \pm\alpha\pi$ generating bijectivity limitations to a, see Figure 3.2. Moreover, Lemma 5.1 tell us that a maps $\mathbb{C} \setminus \overline{\mathbb{D}}$ into $\mathcal{D}(a)$. Let $\rho < \sup\{|a(z)| \colon z \in K_\varepsilon\}$ be a positive constant and consider the regions $S_0 \coloneqq B(0, \rho) \setminus \mathcal{D}(a)$ and $S \coloneqq \mathcal{D}(a) \cap B(0, \rho)$, which we split as follows (see Figure 3.3 right): S_1 is the subset of S enclosed by the curves $\rho\mathbb{T}$, $\mathcal{R}(a)$, and γ_-, including γ_- only; S_2 is the subset of S enclosed by the curves $\rho\mathbb{T}$, $\mathcal{R}(a)$, and γ_+, including γ_+ only; and S_3 is the open subset of S enclosed by the curves $\rho\mathbb{T}$, γ_-, and γ_+. We thus have

$$S = S_1 \cup S_2 \cup S_3.$$

It is easy to see that, for every sufficiently large n, we have no eigenvalues of $T_n(a)$ in S_0. Since $\mathrm{wind}(a - \lambda) = 0$ for each $\lambda \in S_0$, the operator $T(a - \lambda)$ must be invertible and the *Finite Section Method* od Proposition 2.7 is applicable, which means that for every sufficiently large n the matrix $T_n(a - \lambda)$ is invertible and hence, λ is not an eigenvalue of $T_n(a)$. The regions S_1, S_2, and S_3 will be our working sets for λ. In [1] Bogoya, Grudsky and Malysheva raised the following conjecture:

$$\textit{For every sufficiently large } n\textit{, } T_n(a) \textit{ has no eigenvalues in } S_3. \tag{3.4}$$

We will prove this conjecture for the cases $\frac{1}{2} < \alpha < 1$ and $0 < \alpha < \frac{1}{2}$ with $|\arg \lambda| > \frac{\pi}{2}$ (see Theorem 6.6).

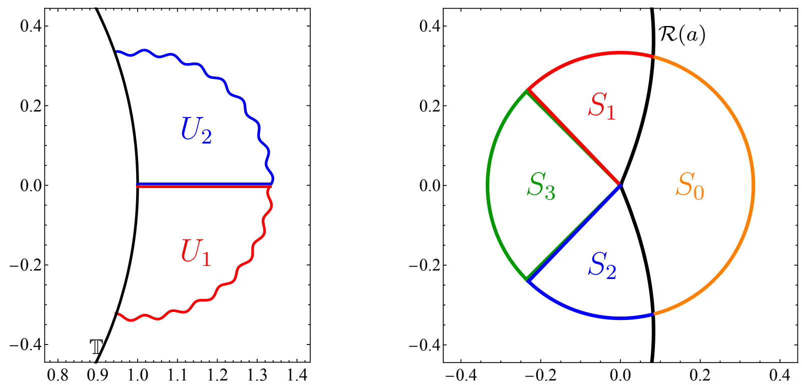


Figure 3.3: (Refer [1]) The bijectivity regions of the symbol a near to the point $t = 1$.

In order to simplify our calculations, throughout the Chapter (5) we use the parameter

$$\Lambda \coloneqq (n+1)\lambda^{\frac{1}{\alpha}} \tag{3.5}$$

divided in to two cases: $m \leqslant |\Lambda| \leqslant M$ for certain constants $0 < m < M < \infty$ (depending only on the symbol a) and $|\Lambda| \to 0$, and the case $|\Lambda| \to \infty$ including the situation where λ is bounded away from zero. Throughout the Chapter (6), let ψ be the argument of λ, δ a small positive constant (see Figure 3.2), and consider the sets

$$R_1 \coloneqq \{\lambda \in S \colon \alpha(\pi - \delta) \leqslant \psi < \alpha\pi\};$$
$$R_2 \coloneqq \{\lambda \in S \colon -\alpha\pi < \psi \leqslant -\alpha(\pi - \delta)\}.$$

The following are main results.

Theorem 3.4 (Refer [1]). *Let a be the symbol (1.2) satisfying the Properties 3.2. A point $\lambda \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$ is an eigenvalue of $T_n(a)$ if and only if there exists numbers m, M (depending only on the symbol a) satisfying $0 < m \leqslant |\Lambda| \leqslant M$, and*

$$\frac{2\pi i}{\alpha} \mathrm{e}^{i\psi(\frac{1}{\alpha}-1)} \mathrm{e}^{\Lambda} = \int_0^\infty \mathrm{e}^{-|\Lambda| v} b(v,\psi)\, \mathrm{d}v + \Delta_1(\lambda, n),$$

where

$$b(v,\psi) \coloneqq \frac{\mathrm{e}^{-i\alpha\pi}}{v^\alpha - \mathrm{e}^{i(\psi - \alpha\pi)}} - \frac{\mathrm{e}^{i\alpha\pi}}{v^\alpha - \mathrm{e}^{i(\psi+\alpha\pi)}},$$

$\psi = \arg\lambda$, and $\Delta_1(\mu, n)$ is a function defined for $\mu \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$ and satisfying $\Delta_1(\mu, n) = O\Big(\frac{1}{n^\alpha}\Big)$ as $n \to \infty$ uniformly in μ.

The previous Theorem 3.4 is important when doing numerical experiments, but using a change of variable and doing some rotations, we can re-write it in terms of Λ alone with the disadvantage of having complex integration paths.

Corollary 3.5 (Refer [1]). *Let a be the symbol (1.2) satisfying the Properties 3.2. A point $\lambda \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$ is an eigenvalue of $T_n(a)$ if and only if there exists positive numbers m, M (depending only on the symbol a) satisfying $m \leqslant |\Lambda| \leqslant M$, and*

$$\frac{2\pi i}{\alpha}\mathrm{e}^{\Lambda} = \int_C \mathrm{e}^{-\Lambda u}\beta(u)\,\mathrm{d}u + \Delta_2(\lambda, n),$$

where

$$\beta(u) := \frac{1}{u^{\alpha}\mathrm{e}^{i\alpha\pi} - 1} - \frac{1}{u^{\alpha}\mathrm{e}^{-i\alpha\pi} - 1},$$

the integration path C is the straight line from 0 to $\infty\mathrm{e}^{-i\frac{3}{4}\pi}$ if $\lambda \in S_1$ or the straight line from 0 to $\infty\mathrm{e}^{i\frac{3}{4}\pi}$ if $\lambda \in S_2$, and $\Delta_2(\mu, n)$ is a function which is defined for any $\mu \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$ and satisfies $\Delta_2(\mu, n) = O\left(\frac{1}{n^{\alpha}}\right)$ as $n \to \infty$ uniformly in μ.

Remember that there are many ways to go to infinity in the complex plane. The symbol $\infty\mathrm{e}^{-i\frac{3}{4}\pi}$ means that we go to infinity in the direction of the argument of $\frac{3}{4}\pi$, and similarly for $\infty\mathrm{e}^{i\frac{3}{4}\pi}$

To get the eigenvalues of $T_n(a)$ from the previous corollary we proceed as follows. Consider the function

$$F(\Lambda) := \frac{2\pi i}{\alpha}\mathrm{e}^{\Lambda} - \int_C \mathrm{e}^{-\Lambda u}\beta(u)\,\mathrm{d}u, \tag{3.6}$$

where C and β are as in Corollary 3.5. Consider the complex sets

$$\hat{S}_\ell := \{\Lambda = (n+1)\lambda^{\frac{1}{\alpha}} \colon \lambda \in S_\ell \quad \text{and} \quad m \leqslant |\Lambda| \leqslant M\} \quad (\ell = 1, 2, 3).$$

For each sufficiently large n, the function F is analytic in $\hat{S}_1 \cup \hat{S}_2$. We can think of Δ_2 as a function of Λ with parameter n which, for each sufficiently large n, will be analytic in $\hat{S}_1 \cup \hat{S}_2$ also. Let $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ be the eigenvalues of $T_n(a)$, then according to Corollary 3.5, if $\lambda_j^{(n)} \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$, the corresponding $\Theta_j^{(n)} := (n+1)(\lambda_j^{(n)})^{\frac{1}{\alpha}} \in \hat{S}_1 \cup \hat{S}_2$ will be a zero of $F(\cdot) - \Delta_2(\cdot, n)$.

Theorem 3.6 (Refer [1]). *Under the same assumptions as in Theorem 3.4, consider the function F in (3.6) and suppose that Λ_j $(1 \leqslant j \leqslant k)$ are its roots located in $\hat{S}_1 \cup \hat{S}_2$ with $F'(\Lambda_j) \neq 0$ for each j. We then have*

$$\lambda_j^{(n)} = \left(\frac{\Lambda_j}{n+1}\right)^{\alpha} (1 + \Delta_3(\Lambda_j, n)),$$

where $\Delta_3(\Lambda, n) = O\left(\frac{1}{n^{\alpha}}\right)$ as $n \to \infty$ uniformly in Λ.

The previous theorem gives us a simple method to get the extreme eigenvalues of $T_n(a)$. To approximate $\lambda_j^{(n)}$, for every sufficiently large n, we only need to calculate numerically (see Table 4.2) the extreme zeros Λ_j of F once.

Chapter 4

A Key Example

The symbol studied by Dai, Geary, and Kadanoff [3] was

$$a(t) = \left(2 - t - \frac{1}{t}\right)^{\gamma} (-t)^{\beta} = (-1)^{3\gamma+\beta} t^{\beta-\gamma} (1-t)^{2\gamma},$$

where $0 < \gamma < -\beta < 1$. In the case $\beta = \gamma - 1$, this function a becomes our symbol with $\alpha = 2\gamma$, we omit the constant $(-1)^{4\gamma-1}$, because it is just a rotation. The conjecture of [3] is that

$$t_{\lambda_{j,n}} \sim n^{(2\gamma+1)n^{-1}} \exp\left(\frac{-2\pi i j}{n}\right).$$

Consider the symbol

$$a(t) = \frac{1}{t}(1-t)^{\frac{3}{4}} \quad (t \in \mathbb{T}).$$

4.1 Inner eigenvalues

Expansions (3.2) and (3.3) prove this conjecture when λ is bounded away from zero, giving us an error bound and a mathematical justification.

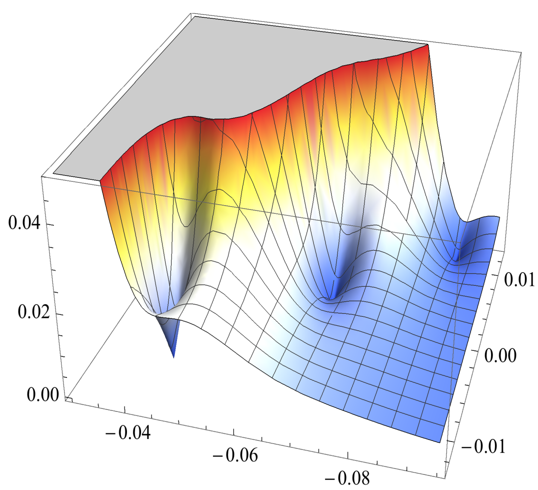
The results are valid outside a small open neighborhood W_0 of the origin, now we take $W_0 = B_{1/5}(0)$ be the disk of radius $\frac{1}{5}$ centered at zero. Table 4.1 shows the data of numerical computations. It reveals that the maximum error of (3.2) with one term is reduced by nearly 10 times when considering the second term; see also Figure 3.1.

n	256	512	1024	2048	4096
(3.2) with 1 term	1.6×10^{-2}	8.1×10^{-3}	4.1×10^{-3}	2.1×10^{-3}	1.0×10^{-3}
(3.2) with 2 terms	1.7×10^{-3}	4.5×10^{-4}	1.2×10^{-4}	3.2×10^{-5}	8.7×10^{-6}
(3.3) with 1 term	5.1×10^{-2}	2.8×10^{-2}	1.5×10^{-2}	8.3×10^{-3}	4.4×10^{-3}
(3.3) with 2 terms	1.5×10^{-2}	7.9×10^{-3}	4.1×10^{-3}	2.1×10^{-3}	1.0×10^{-3}
(3.3) with 3 terms	1.4×10^{-3}	4.3×10^{-4}	1.3×10^{-4}	3.7×10^{-5}	1.1×10^{-5}

Table 4.1: (Refer [2]) The table shows the maximum error obtained with those different formulas for the eigenvalues of $T_n\big(\frac{1}{t}(1-t)^{\frac{3}{4}}\big)$ for different values of n. The data was obtained by comparison with the solutions given by *MATLAB*, taking into account only the eigenvalues with absolute value greater than or equal to $\frac{1}{5}$.

4.2 Extreme eigenvalues

The Theorem 3.4 and Corollary 3.5 show that the conjecture is false when $\lambda \to 0$.



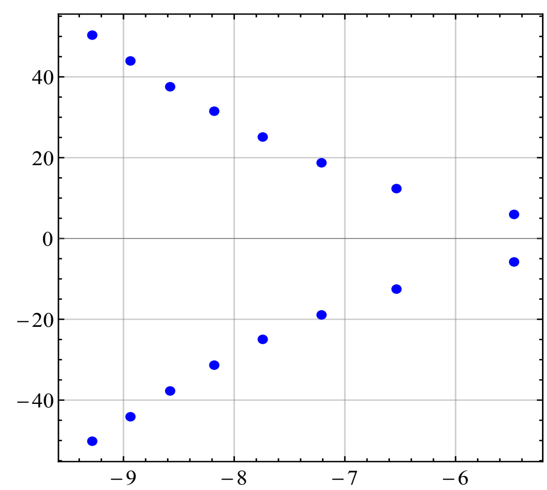


Figure 4.1: (Refer [1]) Left: The norm of $F((n+1)(\cdot)^{\frac{1}{\alpha}})$ for $n = 512$. We see 3 zeros corresponding to 3 consecutive extreme eigenvalues. Right: the 16 zeros of F closest to zero.

In order to approximate the extreme eigenvalues of $T_n(a)$, we worked with the function F in Theorem 3.6. See Figure 4.1.

The symbol $a(t) = \frac{1}{t}(1-t)^{\frac{3}{4}} \quad (t \in \mathbb{T})$ satisfies the properties 1–3 with $\alpha = \frac{3}{4}$ and $f(t) = 1$. According to [19] the eigenvalues of $T_n(a)$ must approximate (in the Hausdorff metric) $\mathcal{R}(a)$ as n increases. See Figure 4.2 (right). In this case the Fourier coefficients can be calculated exactly as $a_k = (-1)^{k+1}\binom{3/4}{k}$.

$-5.46821203708560140608242019410002 \pm 5.79836828171488882078964590677784\,i$
$-6.53144282368424268303380893719 26 \pm 12.36752874007455479774251838295 9\,i$
$-7.21469025247000291423761345061 39 \pm 18.76672662227751957530356959243 3\,i$
$-7.73918325742776483484401500306 17 \pm 25.10704781796458343661411839918 4\,i$
$-8.18017206797400425759920125374 52 \pm 31.41906593601632747585381948555 6\,i$
$-8.57272231175807078598177447378 71 \pm 37.71493429517464942469416572416 6\,i$
$-8.93608902953694661704275307385 61 \pm 44.00051894433324861111087237244 8\,i$
$-9.28200063350184683571769904946 08 \pm 50.27902156031815041271340542618 1\,i$

Table 4.2: (Refer [1]) The 16 zeros of F closest to zero with 32 decimal places (see Figure 4.1 right).

Let $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ be the eigenvalues of $T_n(a)$ numbered counterclockwise starting from the closest one to zero with positive imaginary part. See Figure 4.2 (left). Note that when f is real-valued, then the eigenvalues of $T_n(a)$ as well as the zeros of F come in conjugated pairs. Let $\Lambda_1, \ldots, \Lambda_n$ be the zeros of F, and take

$$\hat{\lambda}_j^{(n)} := \left(\frac{\Lambda_j}{n+1}\right)^{\alpha} \quad (j = 1, \ldots, n)$$

be the approximated eigenvalues obtained from the zeros of F. Finally let $\varepsilon_j^{(n)}$ and $\hat{\varepsilon}_j^{(n)}$ be our individual and relative individual errors, respectively, i.e.

$$\varepsilon_j^{(n)} := |\lambda_j^{(n)} - \hat{\lambda}_j^{(n)}| \quad \text{and} \quad \hat{\varepsilon}_j^{(n)} := \frac{|\lambda_j^{(n)} - \hat{\lambda}_j^{(n)}|}{|\lambda_j^{(n)}|}.$$

See Figure 4.3 and Tables 4.3 and 4.4. The data was obtained with *Wolfram Mathematica*.

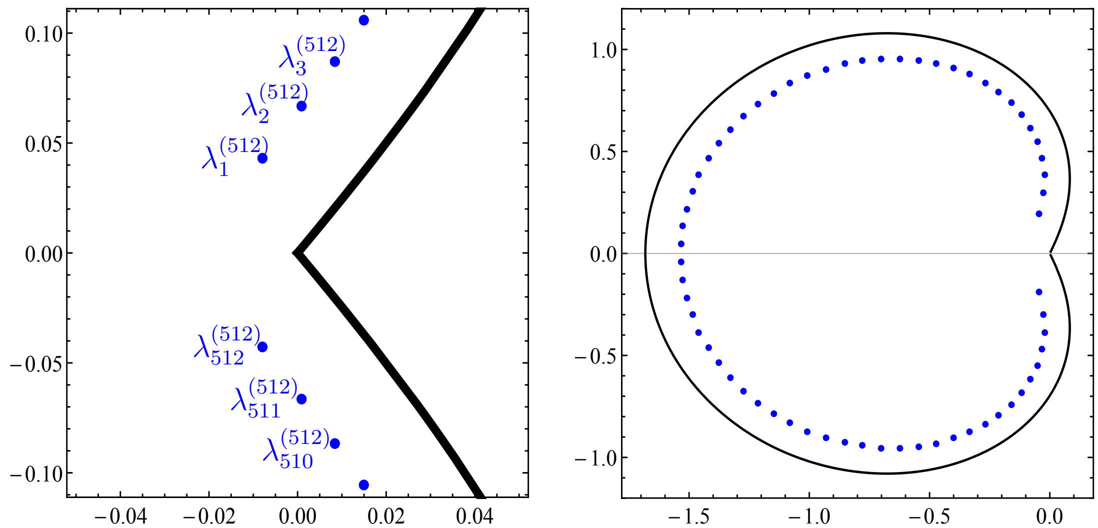


Figure 4.2: (Refer [1]) Range of $a(t) = \frac{1}{t}(1-t)^{\frac{3}{4}}$ (black curve) and spectrum of $T_n(a)$ (blue dots) for $n = 512$ (left) and $n = 64$ (right).

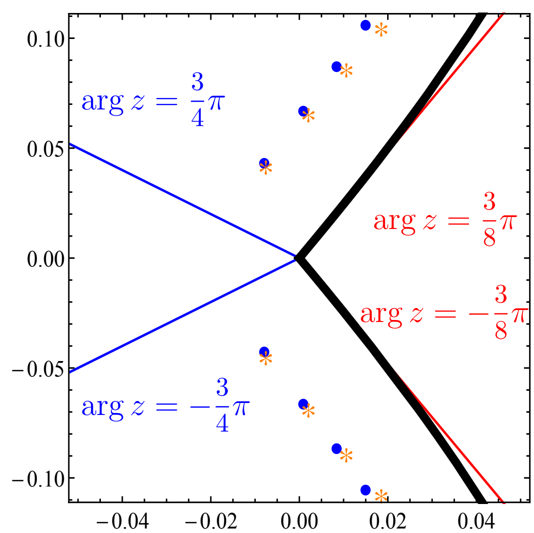


Figure 4.3: (Refer [1]) Range of $a(t) = \frac{1}{t}(1-t)^{\frac{3}{4}}$ (black curve), a few extreme exact and approximated eigenvalues $\lambda_j^{(512)}$ (blue dots) and $\hat{\lambda}_j^{(512)}$ (orange stars), respectively.

In the following chapters we are going to complete the details of prove the results used in this chapters (Refer [1] and [2]).

n	128	256	512	1024	2048	4096
$\varepsilon_1^{(n)}$	$4.39\cdot 10^{-3}$	$1.28\cdot 10^{-3}$	$3.58\cdot 10^{-4}$	$9.16\cdot 10^{-5}$	$1.82\cdot 10^{-5}$	$1.79\cdot 10^{-6}$
$\varepsilon_2^{(n)}$	$1.20\cdot 10^{-2}$	$3.51\cdot 10^{-3}$	$9.81\cdot 10^{-4}$	$2.51\cdot 10^{-4}$	$4.99\cdot 10^{-5}$	$4.92\cdot 10^{-6}$
$\varepsilon_3^{(n)}$	$2.28\cdot 10^{-2}$	$6.66\cdot 10^{-3}$	$1.86\cdot 10^{-3}$	$4.77\cdot 10^{-4}$	$9.48\cdot 10^{-5}$	$9.36\cdot 10^{-6}$
$\varepsilon_4^{(n)}$	$3.64\cdot 10^{-2}$	$1.07\cdot 10^{-2}$	$2.99\cdot 10^{-3}$	$7.65\cdot 10^{-4}$	$1.52\cdot 10^{-4}$	$1.50\cdot 10^{-5}$
$\varepsilon_5^{(n)}$	$5.27\cdot 10^{-2}$	$1.55\cdot 10^{-2}$	$4.33\cdot 10^{-3}$	$1.11\cdot 10^{-3}$	$2.20\cdot 10^{-4}$	$2.18\cdot 10^{-5}$
$\varepsilon_6^{(n)}$	$7.16\cdot 10^{-2}$	$2.10\cdot 10^{-2}$	$5.89\cdot 10^{-3}$	$1.51\cdot 10^{-3}$	$3.00\cdot 10^{-4}$	$2.96\cdot 10^{-5}$
$\varepsilon_7^{(n)}$	$9.29\cdot 10^{-2}$	$2.73\cdot 10^{-2}$	$7.65\cdot 10^{-3}$	$1.96\cdot 10^{-3}$	$3.89\cdot 10^{-4}$	$3.84\cdot 10^{-5}$
$\varepsilon_8^{(n)}$	$1.16\cdot 10^{-1}$	$3.43\cdot 10^{-2}$	$9.61\cdot 10^{-3}$	$2.46\cdot 10^{-3}$	$4.89\cdot 10^{-4}$	$4.83\cdot 10^{-5}$

Table 4.3: (Refer [1]) The error $\varepsilon_j^{(n)}$ for the 8 eigenvalues of $T_n(a)$ closest to zero and with positive imaginary part. Here $a(t) = \frac{1}{t}(1-t)^{\frac{3}{4}}$.

n	128	256	512	1024	2048	4096
$\hat{\varepsilon}_1^{(n)}$	$3.63\cdot 10^{-2}$	$1.71\cdot 10^{-2}$	$8.18\cdot 10^{-3}$	$3.50\cdot 10^{-3}$	$1.17\cdot 10^{-3}$	$1.94\cdot 10^{-4}$
$\hat{\varepsilon}_2^{(n)}$	$6.54\cdot 10^{-2}$	$3.16\cdot 10^{-2}$	$1.47\cdot 10^{-2}$	$6.31\cdot 10^{-3}$	$2.10\cdot 10^{-3}$	$3.48\cdot 10^{-4}$
$\hat{\varepsilon}_3^{(n)}$	$9.48\cdot 10^{-2}$	$4.58\cdot 10^{-2}$	$2.13\cdot 10^{-2}$	$9.13\cdot 10^{-3}$	$3.04\cdot 10^{-3}$	$5.05\cdot 10^{-4}$
$\hat{\varepsilon}_4^{(n)}$	$1.24\cdot 10^{-1}$	$6.00\cdot 10^{-2}$	$2.80\cdot 10^{-2}$	$1.20\cdot 10^{-2}$	$3.99\cdot 10^{-3}$	$6.62\cdot 10^{-4}$
$\hat{\varepsilon}_5^{(n)}$	$1.54\cdot 10^{-1}$	$7.43\cdot 10^{-2}$	$3.46\cdot 10^{-2}$	$1.48\cdot 10^{-2}$	$4.94\cdot 10^{-3}$	$8.19\cdot 10^{-4}$
$\hat{\varepsilon}_6^{(n)}$	$1.84\cdot 10^{-1}$	$8.87\cdot 10^{-2}$	$4.13\cdot 10^{-2}$	$1.77\cdot 10^{-2}$	$5.89\cdot 10^{-3}$	$9.77\cdot 10^{-4}$
$\hat{\varepsilon}_7^{(n)}$	$2.14\cdot 10^{-1}$	$1.03\cdot 10^{-1}$	$4.80\cdot 10^{-2}$	$2.05\cdot 10^{-2}$	$6.84\cdot 10^{-3}$	$1.13\cdot 10^{-3}$
$\hat{\varepsilon}_8^{(n)}$	$2.44\cdot 10^{-1}$	$1.17\cdot 10^{-1}$	$5.47\cdot 10^{-2}$	$2.34\cdot 10^{-2}$	$7.80\cdot 10^{-3}$	$1.29\cdot 10^{-3}$

Table 4.4: (Refer [1]) Relative individual error $\hat{\varepsilon}_j^{(n)}$ for the 8 eigenvalues of $T_n(a)$ closest to zero and with positive imaginary part. We worked here with $a(t) = \frac{1}{t}(1-t)^{\frac{3}{4}}$.

Chapter 5

Behavior of Inner Eigenvalues

5.1 Toeplitz determinant

Lemma 5.1 (Refer [2]). *Let a be the symbol satisfying the Properties 3.1. Then, for each $\lambda \in \mathcal{D}(a)$ and every $n \in \mathbb{N}$, and with $[\,\cdot\,]_n$ denoting the nth Fourier coefficient,*

$$D_n(a-\lambda) = (-1)^n h_0^{n+1} \left[\frac{1}{h(t)-\lambda t}\right]_n. \tag{5.1}$$

Proof. We get the entries of the matrices $T_n(a-\lambda)$ and $T_{n+1}(h-\lambda t)$, and the relationship between them. For $k \in \mathbb{N}$ we have

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(\mathrm{e}^{i\theta})\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} h(\mathrm{e}^{it\theta})\mathrm{e}^{i\theta(k+1)}\,\mathrm{d}\theta = h_{k+1},$$

$$\big[h(t)-\lambda t\big]_k = h_k - \frac{\lambda}{2\pi}\int_0^{2\pi} \mathrm{e}^{i\theta(1-k)}\,\mathrm{d}\theta = \begin{cases} h_k - \lambda, & k = 1; \\ h_k, & k \neq 1. \end{cases}$$

Remember that $T_n(a-\lambda)$ and $T_{n+1}(h-\lambda t)$ are Hessenberg matrices, thus $a_{-k} = 0$

$\forall k \geq 0$. So

$$T_{n+1}(h-\lambda t) = \left[\begin{array}{ccccc|c} h_0 & 0 & 0 & \cdots & 0 & 0 \\ \hline h_1-\lambda & h_0 & 0 & \cdots & 0 & 0 \\ h_2 & h_1-\lambda & h_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 & 0 \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1-\lambda & h_0 \end{array}\right] \tag{5.2}$$

and

$$T_n(a-\lambda) = \begin{bmatrix} h_1-\lambda & h_0 & 0 & \cdots & 0 \\ h_2 & h_1-\lambda & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1-\lambda \end{bmatrix}.$$

Note that $T_{n+1}(h-\lambda t)$ is non-singular, because $h_0 \neq 0$, and applying the Cramer's rule to the system $AX = B$ for $A = T_{n+1}(h-\lambda t)$, $X = A^{-1}$, and $B = I_{n+1}$, we can take the 1st row and $(n+1)$th column to get:

$$\left[T_{n+1}^{-1}(h-\lambda t)\right]_{(n+1,1)} = (-1)^{n+2}\frac{D_n(a-\lambda)}{D_{n+1}(h-\lambda t)}. \tag{5.3}$$

We claim that $h(t)-\lambda t$ is invertible in H^∞. To see this, we must show that $h(t) \neq \lambda t$ for all $t \in \overline{\mathbb{D}}$ and each $\lambda \in \mathcal{D}(a)$. Let λ be a point in $\mathcal{D}(a)$. For each $t \in \mathbb{T}$ we have $h(t) \neq \lambda t$ because $\lambda \notin \partial\mathcal{D}(a) = \mathcal{R}(a)$. By assumption, $\operatorname{wind}_\lambda(a) = -1$ for $\lambda \in \mathcal{D}(a)$, as $\mathcal{R}(a-\lambda)$ is a translation of $\mathcal{R}(a)$ thus

$$\begin{aligned} \operatorname{wind}_0(a) = -1 = \operatorname{wind}_0(a-\lambda) &= \operatorname{wind}_0\left(\frac{1}{t}h(t)-\lambda\right) = \operatorname{wind}_0\left(\frac{1}{t}(h(t)-\lambda t)\right) \\ &= \operatorname{wind}_0\left(\frac{1}{t}\right) + \operatorname{wind}_0\big(h(t)-\lambda t\big) = -1 + \operatorname{wind}_0\big(h(t)-\lambda t\big). \end{aligned}$$

It follows that $\operatorname{wind}_0\big(h(t)-\lambda t\big) = 0$, which means that the origin does not belong to the inside domain of the curve $\{h(t)-\lambda t \colon t \in \mathbb{T}\}$. As $h \in H^\infty$, this shows that $h(t) \neq \lambda t$ for all $t \in \mathbb{D}$ and proves our claim. Using the Proposition 2.6, we have that if b is invertible in H^∞, then $T_{n+1}^{-1}(b) = T_{n+1}(1/b)$. Thus, the $(n+1, 1)$ entry of the matrix $T_{n+1}^{-1}\big(h(t) - \lambda t\big)$ is in fact the nth Fourier coefficient of $\big(h(t) - \lambda t\big)^{-1}$,

$$\left[T_{n+1}^{-1}\big(h(t) - \lambda t\big)\right]_{(n+1,1)} = \left[\frac{1}{h(t) - \lambda t}\right]_n.$$

Inserting this in (5.3) we get

$$D_n(a - \lambda) = (-1)^{n+2} D_{n+1}\big(h(t) - \lambda t\big) \left[\frac{1}{h(t) - \lambda t}\right]_n = (-1)^n h_0^{n+1} \left[\frac{1}{h(t) - \lambda t}\right]_n,$$

which completes the proof. □

Expression (5.1) says that the determinant $D_n(a - \lambda)$ can be expressed as the Fourier integral

$$D_n(a - \lambda) = (-1)^n h_0^{n+1} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{-in\theta}}{h\big(\mathrm{e}^{i\theta}\big) - \lambda \mathrm{e}^{i\theta}} \frac{\mathrm{d}\theta}{2\pi},$$

which is our starting point to find an asymptotic expansion for the eigenvalues of $T_n(a)$. There are two major contributions to this integral. The first comes from λ, when it is close to $\mathcal{R}(a)$, and the second results from the singularity at the point 1. We will analyze them in separate sections.

5.2 Contribution of $\boldsymbol{\lambda}$ to the asymptotic behavior of $\boldsymbol{D_n}$

Defining

$$b(z, \lambda) := \frac{1}{h(z) - \lambda z},$$

we have

$$b_n(\lambda) = \int_{-\pi}^{\pi} b\big(\mathrm{e}^{i\theta}, \lambda\big) \mathrm{e}^{-in\theta} \frac{\mathrm{d}\theta}{2\pi}. \tag{5.4}$$

From (5.1) we conclude that

$$D_n(a - \lambda) = (-1)^n h_0^{n+1} b_n(\lambda). \tag{5.5}$$

Lemma 5.2 (Refer [2]). *Let a be the symbol satisfying the Properties 3.1, such that $\mathcal{R}(a)$ is a Jordan curve in $\mathbb{C}$. Let W_0 be a small open neighborhood of zero in $\mathbb{C}$. Assume that h has an analytic extension to an open neighborhood W of $\mathbb{T}\setminus\{1\}$ in $\mathbb{C}$ not containing the point 1 and that $a'(t) \neq 0$ for every $t \in \mathbb{T}\setminus\{1\}$. Then, for each $\lambda \in \mathcal{D}(a)\setminus W_0$ sufficiently close to $\mathcal{R}(a)$, there exists a unique point t_λ in $W \setminus \overline{\mathbb{D}}$ such that $a(t_\lambda) = \lambda$. Moreover, the point t_λ is a simple pole for b.*

Proof. Without loss of generality, we may assume that the extension of a to W is bounded. As $h \in H^\infty$, this extension must map $W \setminus \overline{\mathbb{D}}$ to $\mathcal{D}(a) \cap a(W)$ so $\operatorname{wind}_\lambda(a) \neq 0$. As the range of a has no loops, we have $a'(t) \neq 0$ for all $t \in \mathbb{T}$.

Consider the set $S := \{t \in \mathbb{T} \colon a(t) \notin W_0\}$ we will show that it is compact. We know that $\mathbb{T}$ is compact, then $a(\mathbb{T})$ is compact, now $\mathcal{R}(a)$ is closed thus $\mathcal{R}(a)\backslash W_0$ is closed, furthermore S is compact.

For every $t \in S$, there exists an open neighborhood V_t of t in $\mathbb{C}$ with $V_t \subset W$ such that $a'(t) \neq 0$ for each $t \in V_t$. Thus, there is an open set U_t such that $t \in U_t \subset V_t$ and a is a conformal map (and hence bijective) from U_t to $a(U_t)$. As S is compact, we can take a finite sub-cover from $\{U_t\}_{t\in S}$, say $U := \bigcup_{i=1}^{M} U_{t_i}$. It follows that a is a conformal map (and hence bijective) from $U \supset S$ to $a(U) \supset a(S)$; see Figure 5.1. The lemma then, holds for every $\lambda \in a(U) \cap (\mathcal{D}(a) \setminus W_0)$. Finally, since $a'(t_\lambda) \neq 0$, the point t_λ must be a simple pole of b, moreover

$$\lim_{t\to t_\lambda} \frac{t - t_\lambda}{t(a(t) - \lambda)} = \lim_{t\to t_\lambda} \frac{1}{(a(t) - \lambda) + t a'(t)} = \frac{1}{t_\lambda a'(t_\lambda)} \neq 0.$$

$\square$

Now using that t_λ is a simple pole of b, we split b as follows:

$$b(z,\lambda) = \frac{1}{z(a(z) - \lambda)} = \frac{1}{t_\lambda a'(t_\lambda)(z - t_\lambda)} + f_0(z,\lambda). \tag{5.6}$$

Here f_0 is analytic with respect to z in W and uniformly bounded with respect to λ in $a(W)\setminus W_0$. We calculate the Fourier coefficients of the first term in (5.6)

$$b_n(\lambda) = \frac{1}{t_\lambda a'(\lambda)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{-in\theta}}{(\mathrm{e}^{i\theta} - t_\lambda)} \,\mathrm{d}\theta + \mathcal{I}$$

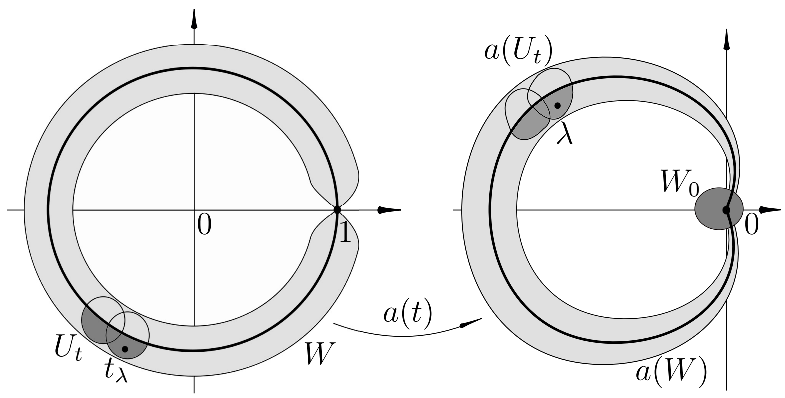


Figure 5.1: (Refer [2]) The map a over the unit circle.

where

$$\mathcal{I} := \int_{-\pi}^{\pi} f_0\big(\mathrm{e}^{i\theta}, \lambda\big)\mathrm{e}^{-in\theta}\frac{\mathrm{d}\theta}{2\pi}.$$

Then

$$b_n(\lambda) = \frac{-1}{t_\lambda^{n+2}a'(t_\lambda)} + \mathcal{I}. \tag{5.7}$$

The first term in (5.7) times $(-1)^n h_0^{n+1}$ is the contribution of t_λ to the asymptotic expansion of $D_n(a-\lambda)$; see (5.5). The function f_0 has a singularity at $z = 1$ and we use this fact to expand $\mathcal{I}$ in the following Section.

5.3 Contribution of 1 to the asymptotic behavior of D_n

In this Section, we will show that the value of $\mathcal{I}$ in (5.7) depends mainly on the singularity at the point 1. Let us write $b(\theta, \lambda)$ and $f_0(\theta, \lambda)$ instead of $b\big(\mathrm{e}^{i\theta}, \lambda\big)$ and $f_0\big(\mathrm{e}^{i\theta}, \lambda\big)$, respectively. Let $\{\phi_1, \phi_2\}$ be a smooth partition of unity over the segment $[-\pi, \pi]$, see Figure 5.2, which means that $\phi_1, \phi_2 \in C^\infty[-\pi, \pi]$, $\phi_1(\theta) + \phi_2(\theta) = 1$ for all $\theta \in [-\pi, \pi]$, the support of ϕ_1 is contained in $[-\pi, -\varepsilon] \cup [\varepsilon, \pi]$, and the support of ϕ_2 is in $[-\delta, \delta]$, where $0 < \varepsilon < \delta$ are small constants. By pasting segments $[-\pi, \pi]$ in both directions, we can continue ϕ_1 and ϕ_2 to the entire real line $\mathbb{R}$, and we will think of these two functions in

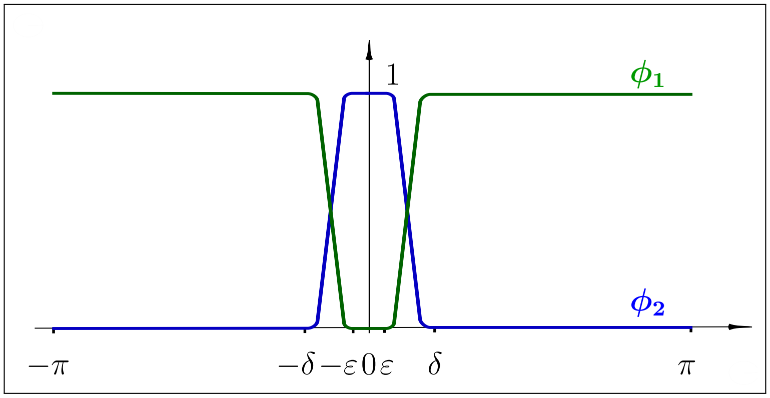


Figure 5.2: (Refer [2]) Partition of unity over the segment $[-\pi, \pi]$

that way.

Lemma 5.3 (Refer [2])**.** *For every sufficiently small positive δ, we have*

$$\mathcal{I} = \int_{-\delta}^{\delta} \phi_2(\theta) b(\theta, \lambda) \mathrm{e}^{-in\theta} \frac{\mathrm{d}\theta}{2\pi} + Q_1(n, \lambda), \tag{5.8}$$

where $Q_1(n, \lambda) = O\Big(\frac{1}{n^\infty}\Big)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Proof. Using the partition of unity $\{\phi_1, \phi_2\}$, we write $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$ where

$$\mathcal{I}_1 := \int_{\varepsilon}^{2\pi-\varepsilon} \phi_1(\theta) f_0(\theta, \lambda) \mathrm{e}^{-in\theta} \frac{\mathrm{d}\theta}{2\pi}, \quad \mathcal{I}_2 := \int_{-\delta}^{\delta} \phi_2(\theta) f_0(\theta, \lambda) \mathrm{e}^{-in\theta} \frac{\mathrm{d}\theta}{2\pi}.$$

The function $\phi_1(\theta) f_0(\theta, \lambda)$ belongs to $C^\infty[\varepsilon, 2\pi - \varepsilon]$. Using integration by parts m-times and taking $q(\theta, \lambda) = \phi_1(\theta) f_0(\theta, \lambda)$ we have

$$\mathcal{I}_1 = \sum_{s=0}^{m-1} (-1)^s \left[\frac{q(\theta, \lambda)^{(s)} (-1)^{s+1} \mathrm{e}^{-in\theta}}{(in)^{s+1}} \right]_{\varepsilon}^{2\pi-\varepsilon} + (-1)^m \int_{\varepsilon}^{2\pi-\varepsilon} \frac{q(\theta, \lambda)^{(m)} \mathrm{e}^{-in\theta} (-i)^m}{(in)^m} \, \mathrm{d}\theta.$$

Because of Riemann Lebesgue's lemma (2.11), this equals

$$\mathcal{I}_1 = \sum_{s=0}^{m-1} O\Big(\frac{1}{n^{s+1}}\Big) + \frac{1}{n^m} o(1).$$

The predominant order is the one with higher degree thus $\mathcal{I}_1 = O\Big(\frac{1}{n^m}\Big)$. We obtain that $\mathcal{I}_1 = O\Big(\frac{1}{n^\infty}\Big)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Solving for $f_0(\theta,\lambda)$ in (5.6), we arrive at $\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22}$ where

$$\mathcal{I}_{21} := \int_{-\delta}^{\delta} \phi_2(\theta) b(\theta,\lambda) \mathrm{e}^{-in\theta} \frac{\mathrm{d}\theta}{2\pi}, \quad \mathcal{I}_{22} := \frac{-1}{t_\lambda a'(t_\lambda)} \int_{-\delta}^{\delta} \frac{\phi_2(\theta)\mathrm{e}^{-in\theta}}{\mathrm{e}^{i\theta} - t_\lambda} \frac{\mathrm{d}\theta}{2\pi}. \tag{5.9}$$

Once more, the function $\phi_2(\theta)\big(\exp(i\theta) - t_\lambda\big)^{-1}$ belongs to $C^\infty[-\delta,\delta]$, with a similar method we conclude that $\mathcal{I}_{22} = O\Big(\frac{1}{n^\infty}\Big)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$. $\square$

Expression (5.8) says that the value of $\mathcal{I}$ basically depends on the integrand $b(\theta,\lambda)\mathrm{e}^{-in\theta}$ at $\theta = 0$. As we can take δ as small as we desire, we can assume that θ is arbitrarily close to zero. Keeping this idea in mind, we will develop an asymptotic expansion for b. For future Reference, we rewrite (5.8) as

$$\mathcal{I} = \mathcal{I}_{21} + Q_1(n,\lambda), \tag{5.10}$$

where $Q_1(n,\lambda) = O\Big(\frac{1}{n^\infty}\Big)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Lemma 5.4 (Refer [2]). *For every sufficiently small positive δ,*

$$\mathcal{I}_{21} = -\sum_{s=0}^{\infty} \frac{1}{\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) h^s(\theta) \mathrm{e}^{-in\theta}}{\mathrm{e}^{i\theta(s+1)}} \frac{\mathrm{d}\theta}{2\pi}. \tag{5.11}$$

Proof. We have

$$\mathcal{I}_{21} = \int_{-\delta}^{\delta} \phi_2(\theta) b(\theta,\lambda) \mathrm{e}^{-in\theta} \frac{\mathrm{d}\theta}{2\pi}. \tag{5.12}$$

Note that

$$b(\theta,\lambda) = \frac{1}{h(\theta) - \lambda \mathrm{e}^{i\theta}} = \frac{-1}{\lambda \mathrm{e}^{i\theta}} \cdot \frac{1}{1 - \lambda^{-1}\mathrm{e}^{-i\theta} h(\theta)}.$$

Now, since f is bounded, $\big|h(\theta)\big| = \big|1 - \mathrm{e}^{i\theta}\big|^\alpha \big|f(\mathrm{e}^{i\theta})\big| \to 0$ when $\theta \to 0$. So there exists a small positive constant δ such that

$$\Big|\lambda^{-1}\mathrm{e}^{-i\theta} h(\theta)\Big| < 1$$

for every $|\theta| < \delta$. Thus,

$$b(\theta,\lambda) = \frac{-1}{\lambda \mathrm{e}^{i\theta}} \sum_{s=0}^{\infty} \Big(\lambda^{-1}\mathrm{e}^{-i\theta} h(\theta)\Big)^s = -\sum_{s=0}^{\infty} \frac{h^s(\theta)}{\lambda^{s+1}\mathrm{e}^{i\theta(s+1)}} \tag{5.13}$$

for every $|\theta| < \delta$. Inserting (5.13) in (5.12) and considering that $\big|\phi_2(\theta) b(\theta,\lambda)\big| \le 1$, the dominated convergence Theorem 2.4 finishes the proof. $\square$

We will use the notation

$$\mathcal{I}_{21s} := \frac{1}{\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) h^s(\theta) \mathrm{e}^{-in\theta}}{\mathrm{e}^{i\theta(s+1)}} \frac{\mathrm{d}\theta}{2\pi}.$$

Because $\phi_2(\theta)\mathrm{e}^{-i\theta} \in C^\infty[-\delta, \delta]$, using the same idea which proves $\mathcal{I}_1 = O\Big(\frac{1}{n^\infty}\Big)$ we have $\mathcal{I}_{21s}|_{s=0} = O\Big(\frac{1}{n^\infty}\Big)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$. With the previous notation, we can rewrite (5.11) as

$$\mathcal{I}_{21} = -\sum_{s=1}^{\infty} \mathcal{I}_{21s} + Q_2(n, \lambda), \tag{5.14}$$

where $Q_2(n, \lambda) = O\Big(\frac{1}{n^\infty}\Big)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$. Finally we will work with $\mathcal{I}_{21s}$.

Lemma 5.5 (Refer [2]). *Let $h(t) = (1 - t)^\alpha f(t)$ with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $f \in C^\infty(\mathbb{T})$. Then,*

$$\mathcal{I}_{21} = \frac{f(1)\Gamma(\alpha + 1)\sin(\alpha\pi)}{\pi\lambda^2 n^{\alpha+1}} + R_1(n, \lambda), \tag{5.15}$$

where $R_1(n, \lambda) = O\Big(\frac{1}{n^{\alpha+\alpha_0+1}}\Big)$ with $\alpha_0 = \min\{\alpha, 1\}$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Proof. All the order terms in this proof work as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$.

We know that $h(\theta) = (1 - \mathrm{e}^{i\theta}) f(\mathrm{e}^{i\theta}) = (-i\theta)^\alpha v(\theta) f\big(\mathrm{e}^{i\theta}\big)$, where the function v equals $\big(i\theta^{-1}(1 - \mathrm{e}^{i\theta})\big)^\alpha$, the branch of the αth power being the one corresponding to the argument in $(-\pi, \pi]$; note that for every sufficiently small positive δ we have $v \in C^\infty[-\delta, \delta]$ since $v(0) = \lim_{\theta \to 0} \left(\frac{1 - \mathrm{e}^{i\theta}}{i\theta}\right)^\alpha = 1$. Thus,

$$\begin{aligned}
\mathcal{I}_{21s} &= \frac{1}{2\pi\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) h^s(\theta) \mathrm{e}^{-in\theta}}{\mathrm{e}^{i\theta(s+1)}} \,\mathrm{d}\theta \\
&= \frac{1}{2\pi\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) \theta^{s\alpha} v^s(\theta) f^s(\mathrm{e}^{i\theta}) \mathrm{e}^{-in\theta}}{\mathrm{e}^{i\theta(s+1)}} \,\mathrm{d}\theta.
\end{aligned}$$

Using $w(\theta) := (-i)^{\alpha s} \dfrac{\phi_2(\theta) v^s(\theta) f^s\big(\mathrm{e}^{i\theta}\big)}{2\pi\lambda^{s+1}\mathrm{e}^{i\theta(s+1)}}$ and $\beta := \alpha s + 1$, the last integral can be written as

$$
\begin{aligned}
\mathcal{I}_{21s} &= \int_{-\delta}^{\delta} \theta^{\beta-1} w(\theta) \mathrm{e}^{-in\theta}\, \mathrm{d}\theta \\
&= \int_{-\delta}^{0} \theta^{\beta-1} w(\theta) \mathrm{e}^{-in\theta}\, \mathrm{d}\theta + \int_{0}^{\delta} \theta^{\beta-1} w(\theta) \mathrm{e}^{-in\theta}\, \mathrm{d}\theta \\
&= \int_{0}^{\delta} (-\tau)^{\beta-1} w(-\tau) \mathrm{e}^{in\tau} d\tau + \int_{0}^{\delta} \theta^{\beta-1} w(\theta) \mathrm{e}^{-in\theta}\, \mathrm{d}\theta = \mathcal{I}_{21s1} + \mathcal{I}_{21s2},
\end{aligned} \tag{5.16}
$$

where

$$
\mathcal{I}_{21s1} := (-1)^{\beta-1} \int_{0}^{\delta} \theta^{\beta-1} w(-\theta) \mathrm{e}^{in\theta}\, \mathrm{d}\theta, \quad \mathcal{I}_{21s2} := \int_{0}^{\delta} \theta^{\beta-1} w(\theta) \mathrm{e}^{-in\theta}\, \mathrm{d}\theta.
$$

Note that $w(\pm\theta) \in C^{\infty}[0, \delta]$ and $w^{(s)}(\pm\delta) = 0$ for all $s \in \mathbb{N}$ because $\phi_2(\theta) \equiv 0$ in a small neighborhood of $\pm\delta$. Applying Theorem 2.12 to $\mathcal{I}_{21s1}$ and Corollary 2.13 to $\mathcal{I}_{21s2}$, we obtain

$$
\mathcal{I}_{21s1} = \frac{(-1)^{\alpha s} w(0) \Gamma(\alpha s + 1) i^{\alpha s+1}}{n^{\alpha s+1}} + Q_3(s, n, \lambda)
$$

and

$$
\mathcal{I}_{21s2} = \frac{w(0) \Gamma(\alpha s + 1) i^{-\alpha s-1}}{n^{\alpha s+1}} + Q_4(s, n, \lambda), \tag{5.17}
$$

where Q_3 and Q_4 are $O\Big(\frac{1}{n^{\alpha s+2}}\Big)$. Substitution of (5.17) in (5.16) yields

$$
\begin{aligned}
\mathcal{I}_{21s} =& \frac{w(0)\Gamma(\alpha s+1)}{n^{\alpha s+1}} \big(\mathrm{e}^{-i\frac{\pi}{2}(\alpha s+1)} + (\mathrm{e}^{i\pi})^{\alpha s} \mathrm{e}^{i\frac{\pi}{2}(\alpha s+1)}\big) + Q_5(s, n, \lambda) \\
=& \frac{f^(1)\Gamma(\alpha s+1)}{2\pi\lambda^{s+1} n^{\alpha s+1}} i \mathrm{e}^{-\frac{\pi}{2} s\alpha} \big(\mathrm{e}^{i\pi\alpha s} \mathrm{e}^{i\frac{\pi}{2}(\alpha s)} - \mathrm{e}^{-i\frac{\pi}{2}(\alpha s)}\big) + Q_5(s, n, \lambda) \\
=& \frac{f^(1)\Gamma(\alpha s+1)}{2\pi\lambda^{s+1} n^{\alpha s+1}} i \big(\mathrm{e}^{i\pi\alpha s} - \mathrm{e}^{-i\alpha s}\big) + Q_5(s, n, \lambda) \\
=& \frac{-C_s}{\lambda^{s+1} n^{\alpha s+1}} + Q_5(s, n, \lambda)
\end{aligned} \tag{5.18}
$$

where

$$
C_s := \frac{1}{\pi} f^s(1) \Gamma(\alpha s + 1) \sin(\alpha\pi s) \tag{5.19}
$$

and $Q_5(s, n, \lambda) = O\Big(\frac{1}{n^{\alpha s+2}}\Big)$. From (5.14) and (5.18) we obtain

$$
\mathcal{I}_{21} = \frac{C_1}{\lambda^2 n^{\alpha+1}} + O\bigg(\frac{1}{n^{\alpha+2}}\bigg) + O\bigg(\frac{1}{n^{2\alpha+1}}\bigg) = \frac{C_1}{\lambda^2 n^{\alpha+1}} + R_1(n, \lambda),
$$

where $R_1(n, \lambda) = O\Big(\frac{1}{n^{\alpha+\alpha_0+1}}\Big)$ here $\alpha_0 := \min\{\alpha, 1\}$. $\square$

The previous calculation gives us the main asymptotic term for $\mathcal{I}_{21}$. If more terms are needed, say m, we must expand $\mathcal{I}_{21}$ from $\mathcal{I}_{21s}|_{s=1}$ to $\mathcal{I}_{21s}|_{s=m}$ and expand each $\mathcal{I}_{21s}$ to m terms, after which, according to the value of α, we need to select the first m principal terms.

Finally we put all the lemmas together to prove Theorem 3.1.

Proof of Theorem 3.1. The proof of this theorem is a direct application of equations (5.5), (5.7), (5.10), and (5.15).
$D_n(a-\lambda) = (-1)^n (h_0)^{n+1} b_n(\lambda)$ but $b_n(\lambda) = \dfrac{-1}{t_\lambda^{n+2} a'(t_\lambda)} + \mathcal{I}$ and $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$ so

$$D_n(a-\lambda) = (-h_0)^{n+1}\left(\frac{1}{t_\lambda^{n+2} a'(t_\lambda)} - \frac{C_1}{\lambda^2 n^{\alpha+1}} + O\left(\frac{1}{n^{\alpha+\alpha_0+1}}\right)\right).$$

$\square$

5.4 Individual eigenvalues

In order to find the eigenvalues of the matrices $T_n(a)$, we need to solve the equations $D_n(a-\lambda) = 0$. We start this Section by locating the zeros of $D_n(a-\lambda)$.

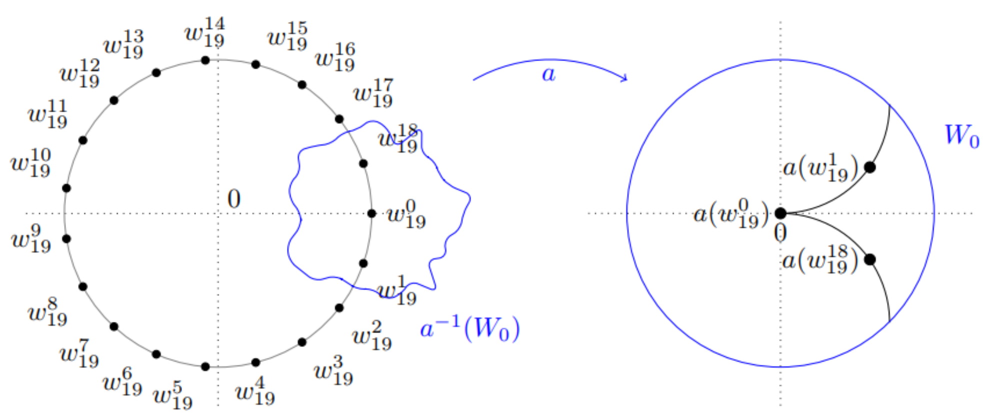


Figure 5.3: 19th root of unity, where $n_1 = n_2 = 1$

Let W_0 be a small open neighborhood of zero in $\mathbb{C}$ and $\omega_n := \exp\left(\frac{-2\pi i}{n}\right)$ nth root of unity, where n is a positive integer. For each n there exist integers n_1 and n_2 such that

$\omega_n^{n_1}$, $\omega_n^{n-n_2} \in a^{-1}(W_0)$ but $\omega_n^{n_1+1}$, $\omega_n^{n-n_2-1} \notin a^{-1}(W_0)$, see Figure (5.3), because W_0 does not cover $a(\mathbb{T})$.

Recall that $\lambda = a(t_\lambda)$. Take an integer j satisfying $n_1 < j < n - n_2$, since $a(t_\lambda)$ and $a'(t_\lambda)$ have no problems in 0 and are analytical, using the relations

$$\frac{1}{t_\lambda^2 a'(t_\lambda)} = \frac{1}{\omega_j^2 a'(\omega_n^j)} + O(|t_\lambda - \omega_n^j|)$$

and

$$\frac{1}{a^2(t_\lambda)} = \frac{1}{a^2(\omega_n^j)} + O(|t_\lambda - \omega_n^j|),$$

where t_λ belongs to a small neighborhood of the point ω_n^j, we see that

$$D_n(a-\lambda) = (-h_0)^{n+1}\Big(\mathcal{T}_1 - \mathcal{T}_2 + \frac{1}{t_\lambda^n} O\big(\big|t_\lambda - \omega_n^j\big|\big) + \frac{1}{n^{\alpha+1}} O\big(\big|t_\lambda - \omega_n^j\big|\big) + Q_6(n, t_\lambda)\Big)$$

$$D_n(a-\lambda) = (-h_0)^{n+1}\Big(\mathcal{T}_1 - \mathcal{T}_2 + O\Big(\Big|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\Big|\Big) + O\Big(\frac{\big|t_\lambda - \omega_n^j\big|}{n^{\alpha+1}}\Big) + Q_6(n, t_\lambda)\Big), \quad (5.20)$$

where $Q_6(n, t_\lambda) = O\Big(\frac{1}{n^{\alpha+\alpha_0+1}}\Big)$ as $n \to \infty$, uniformly with respect to t_λ in $W \setminus a^{-1}(W_0)$, and where t_λ belongs to a small neighborhood of ω_n^j. Here

$$\mathcal{T}_1 := \frac{1}{t_\lambda^n \omega_n^{2j} a'(\omega_n^j)}, \quad \mathcal{T}_2 := \frac{C_1}{a^2(\omega_n^j) n^{\alpha+1}},$$

and $\alpha_0 := \min\{\alpha, 1\}$. Recall C_1 from (5.19). Expression (5.20) makes sense only when t_λ is sufficiently "close" to ω_n^j and thus it is necessary to know whether there exists a zero of $D_n(a-\lambda)$ "close" to ω_n^j. Let

$$t_\lambda = (1+\rho)\exp(i\theta).$$

It is easy to see that $\mathcal{T}_1 - \mathcal{T}_2 = 0$ if and only if $(1+\rho)^n \exp(in\theta) = \dfrac{a^2(\omega_n^j) n^{\alpha+1}}{C_1 \omega_n^{2j} a'(\omega_n^j)}$ then

$$\rho = \left(\frac{|a(\omega_n^j)|^2 n^{\alpha+1}}{|C_1 a'(\omega_n^j)|}\right)^{\frac{1}{n}} - 1 \quad (5.21)$$

and

$$\theta = \theta_j = \frac{1}{n} \arg\left(\frac{a^2(\omega_n^j)}{C_1 \omega_n^{2j} a'(\omega_n^j)}\right) - \frac{2\pi j}{n}$$

for some $j \in \{0, \ldots, n-1\}$. When n tends to infinity, (5.21) shows that ρ remains positive and since $\dfrac{|a(\omega_n^j)|^2}{|C_1 a'(\omega_n^j)|}$ is a bounded constant and $n^{\frac{\alpha+1}{n}} \to 1$ then $\rho \to 0$. The function $\mathcal{T}_1 - \mathcal{T}_2$ has n zeros with respect to $\lambda \in \mathcal{D}(a)$ given by

$$a\big((1+\rho)\mathrm{e}^{i\theta_0}\big), \ldots, \quad a\big((1+\rho)\mathrm{e}^{i\theta_{n-1}}\big).$$

As Lemma 5.2 establishes a 1-1 correspondence between λ and t_λ and the function $D_n(a-\lambda)$ is analytic with respect to λ in $a(W) \setminus W_0$, that is, analytic with respect to t_λ in $W \setminus a^{-1}(W_0)$. We can therefore suppose that $\mathcal{T}_1 - \mathcal{T}_2$ has n zeros with respect to t_λ in the exterior of $\overline{\mathbb{D}}$ given by

$$t_0 := (1+\rho)\mathrm{e}^{i\theta_0}, \ldots, \quad t_{n-1} := (1+\rho)\mathrm{e}^{i\theta_{n-1}}.$$

We take the function "arg" in the interval $(-\pi, \pi]$. Thus, $t_j = (1+\rho)\mathrm{e}^{i\theta_j}$ is the nearest zero to ω_n^j for the Rouché Theorem 2.15 . Consider the neighborhood E_j of t_j sketched in Figure 5.4.

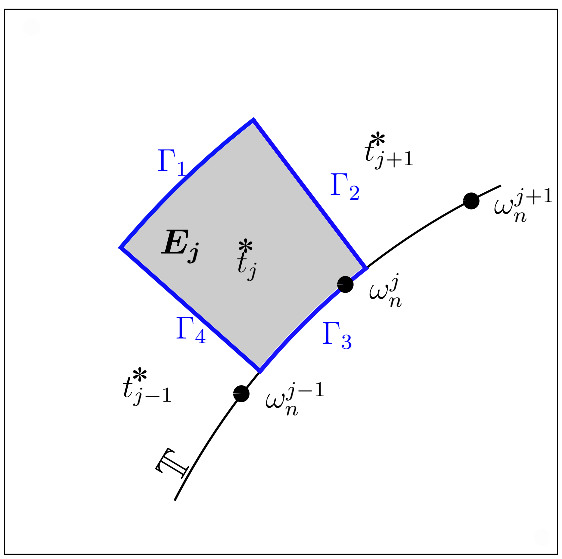
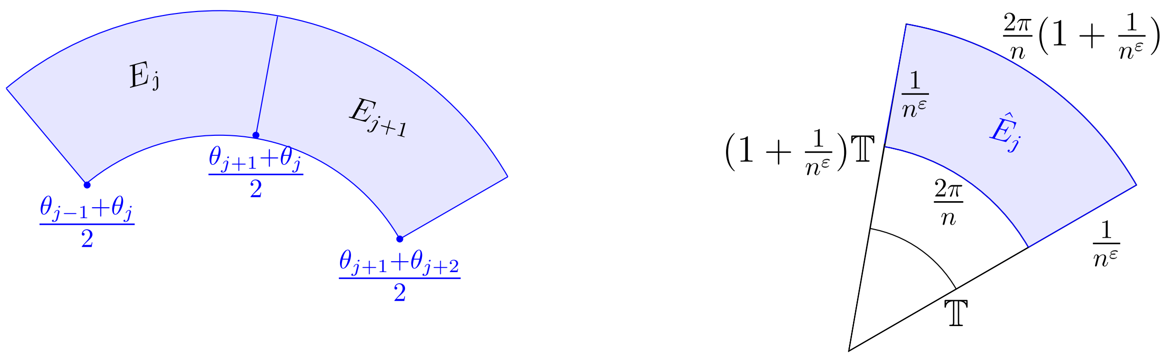


Figure 5.4: (Refer [2]) The neighborhood E_j of t_j in the complex plane.

The boundary of E_j is $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. We have chosen radial segments Γ_2 and Γ_4 so that their length is $\dfrac{1}{n^\varepsilon}$ with $\epsilon \in (0, \alpha_0)$ and all the points in Γ_2 have the common argument $\dfrac{\theta_{j+1}+\theta_j}{2}$, while all the points in Γ_4 have the common argument $\dfrac{\theta_{j-1}+\theta_j}{2}$. As we can see in Figure 5.4, these points run from the unit circle $\mathbb{T}$ to $\left(1+\dfrac{1}{n^{\varepsilon}}\right)\mathbb{T}$. Note also that $\Gamma_1 \subset \left(1+\dfrac{1}{n^{\varepsilon}}\right)\mathbb{T}$ and $\Gamma_3 \subset \mathbb{T}$.



(a) Neighborhoods Ej and E_{j+1}

(b) Neighborhood $\hat{E}_j$

Figure 5.5: Neighborhoods in the complex plane

Theorem 5.6 (Refer [2]). *Let a be the symbol satisfying the Properties 3.1. Let $\epsilon \in (0, \alpha_0)$ be a constant. Then, there exists a family of sets $\{E_j\}_{j=n_1+1}^{n-n_2-1}$ in $\mathbb{C}$ such that*

1. *$\{E_j\}_{j=n_1+1}^{n-n_2-1}$ is a family of pairwise disjoint open sets,*

2. *$\operatorname{diam}(E_j) \leqslant \dfrac{2\pi}{n^{\epsilon}}$,*

3. *$\omega_n^j \in \partial E_j$,*

4. *$D_n\big(a - a(t_\lambda)\big) = D_n(a-\lambda)$ has exactly one zero in each E_j.*

Here $\alpha_0 := \min\{\alpha, 1\}$ and $\operatorname{diam}(E_j) := \sup\{|z_1 - z_2| \colon z_1, z_2 \in E_j\}$.

Proof. 1. It is enough to prove that $E_j \cap E_{j+1} \neq \varnothing$. Because of the definition we have, $\partial E_j \cap \partial E_{j+1} = \{z_0 : \arg(z_0) = \dfrac{\theta_j + \theta_{j+1}}{2}\}$ (see Figure 5.5a), thus $E_j \cap E_{j+1} \neq \varnothing$.

2. Remember that we gave necessary conditions for t_λ be "close" to ω_n^j, this implies that $\arg(t_j) \sim \dfrac{2\pi j}{n}$. Consider the set $\hat{E}_j$ in Figure 5.5b . Thus $\operatorname{diam}(E_j) \sim \operatorname{diam}(\hat{E}_j)$, but $\operatorname{diam}(\hat{E}_j) < \dfrac{2\pi}{n^{\varepsilon}}$ then $\operatorname{diam}(Ej) < \dfrac{2\pi}{n^{\varepsilon}}$.

3. We know that each consecutive pair of nth roots of unity, are separated by an arc with length $\dfrac{2\pi}{n} < \dfrac{2\pi}{n^{\varepsilon}}$, then using the previous item, we have $\omega_n^j \in E_j$, moreover $\omega_n^j \in \Gamma_3$.

4. We prove assertion 4 by studying the behavior of $|D_n(a-\lambda)|$ with respect to $t_\lambda \in \Gamma$.

For $t_\lambda \in \Gamma_1$ is $|t_\lambda|^n = \left|1+\dfrac{1}{n^{\varepsilon}}\right|^n$, using that $\left(1+\dfrac{1}{n^{\epsilon}}\right)^{-n} = \exp\left(-n\log\left(1+\dfrac{1}{n^{\varepsilon}}\right)\right)$. We have as $n\to\infty$,

$$|\mathcal{T}_1|_{\Gamma_1} = \frac{1}{|a'(\omega_n^j)|}\left(1+\frac{1}{n^{\epsilon}}\right)^{-n} = \frac{\exp(-n^{1-\varepsilon})}{|a'(\omega_n^j)|} + O\left(\frac{\exp(-n^{1-\varepsilon})}{n^{2\varepsilon-1}}\right),$$

and using the Taylor's series of log, we get

$$|\mathcal{T}_2|_{\Gamma_1} = \frac{1}{n^{\alpha+1}}\left|\frac{C_1}{a^2(\omega_n^j)}\right|.$$

By a similar argument, we have

$$\left|O\left(\left|\frac{t_\lambda-\omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_1} = O\left(\frac{\exp(-n^{1-\epsilon})}{n^{\epsilon}}\right),$$

also

$$\left|O\left(\left|\frac{t_\lambda-\omega_n^j}{n^{\alpha+1}}\right|\right)\right|_{\Gamma_1} = O\left(\frac{1}{n^{\epsilon+\alpha+1}}\right) \quad \text{and} \quad |Q_6(n,t_\lambda)|_{\Gamma_1} = O\left(\frac{1}{n^{\alpha+\alpha_0+1}}\right).$$

When n goes to infinity, the modulus of $\mathcal{T}_2$ decreases at polynomial speed over Γ_1, while the module of the remaining terms in (5.20) are smaller over Γ_1. Thus,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_1} = \frac{1}{n^{\alpha+1}}\left|\frac{C_1}{a^2(\omega_n^j)}\right| + O\left(\frac{1}{n^{\alpha+\epsilon+1}}\right).$$

For $t_\lambda \in \Gamma_3$ is $|t_\lambda| = 1$. We get, as $n\to\infty$,

$$|\mathcal{T}_1|_{\Gamma_3} = \frac{1}{|a'(\omega_n^j)|} \quad \text{and} \quad |\mathcal{T}_2|_{\Gamma_3} = \frac{1}{n^{\alpha+1}}\cdot\left|\frac{C_1}{a^2(\omega_n^j)}\right|.$$

Note that $|t_\lambda-\omega_n^j| = O\left(\dfrac{1}{n}\right)$ so

$$\left|O\left(\left|\frac{t_\lambda-\omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_3} = O\left(\frac{1}{n}\right), \quad \left|O\left(\frac{|t_\lambda-\omega_n^j|}{n^{\alpha+1}}\right)\right|_{\Gamma_3} = O\left(\frac{1}{n^{\alpha+2}}\right),$$

and $|Q_6(n,t_\lambda)|_{\Gamma_3} = O\Big(\frac{1}{n^{\alpha+\alpha_0+1}}\Big)$. When n goes to infinity, the modulus of $\mathcal{T}_1$ remains constant over Γ_3, while the moduli of the remaining terms in (5.20) are smaller there. Consequently,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_3} = \frac{1}{|a'(\omega_n^j)|} + O\left(\frac{1}{n}\right).$$

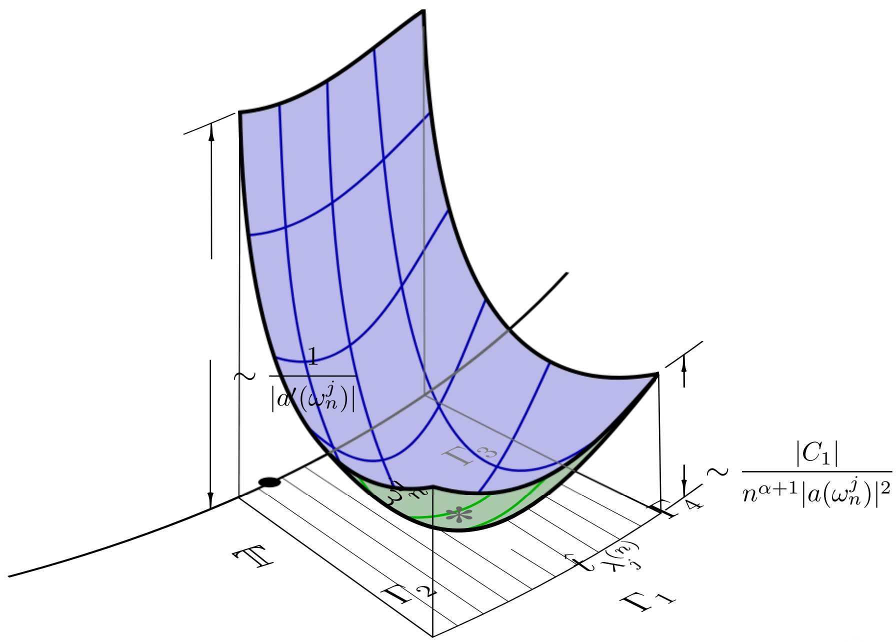


Figure 5.6: (Refer [2]) Function $\dfrac{D_(a-\lambda)}{h_0^{n+1}}$ in neighborhood E_j

For the radial segments Γ_2 and Γ_4, we start by showing that $\mathcal{T}_1$ and $-\mathcal{T}_2$ have the same argument principal there. Since t_j is a zero of $\mathcal{T}_1 - \mathcal{T}_2$, we deduce that

$$\arg\left(\frac{1}{t_j^n \omega_n^{2j} a'(\omega_n^j)}\right) = \arg\left(\frac{C_1}{a^2(\omega_n^j) n^{\alpha+1}}\right),$$

$$\arg\left(\frac{1}{t_j^n}\right) + \arg\left(\frac{1}{\omega_n^{2j} a'(\omega_n^j)}\right) = \arg\left(\frac{C_1}{a^2(\omega_n^j)}\right).$$

Note that $\arg(t_j^{-1}) = \arg(\overline{t_j}) = -\arg(t_j) = -\theta_j$ and "if we sum $2k\pi$ the second argument change the representative of class", thus

$$-n\theta_j + \arg\left(\frac{1}{\omega_n^{2j} a'(\omega_n^j)}\right) = \arg\left(\frac{C_1}{a^2(\omega_n^j)}\right). \tag{5.22}$$

For $t_\lambda \in \Gamma_4$ we have

$$\begin{aligned}\arg(\mathcal{T}_1) &= \arg\left(\frac{1}{t_\lambda^n \omega_n^{2j} a'(\omega_n^j)}\right) = -\frac{n}{2}(\theta_{j-1}+\theta_j) + \arg\left(\frac{1}{\omega_n^{2j} a'(\omega_n^j)}\right)\\ &= \frac{n}{2}(\theta_j - \theta_{j-1}) + \arg\left(\frac{C_1}{a^2(\omega_n^j)}\right) = \pi + \arg\left(\frac{C_1}{a^2(\omega_n^j)}\right) = \arg(-\mathcal{T}_2).\end{aligned}$$

Here the second line is due to $(\theta_j - \theta_{j-1})$ is the angle between t_j and t_{j-1}, note that t_j is uniformly distanced because is a solution when take nth root, then $(\theta_j - \theta_{j-1}) = \frac{2\pi}{n}$. In addition, as $n \to \infty$,

$$\left| O\left(\left|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_4} = O\left(\frac{1}{n^\epsilon |t_\lambda|^n}\right), \quad \left| O\left(\frac{\left|t_\lambda - \omega_n^j\right|}{n^{\alpha+1}}\right)\right|_{\Gamma_4} = O\left(\frac{1}{n^{\alpha+\epsilon+1}}\right),$$

and $|Q_6(n,t_\lambda)|_{\Gamma_4} = O\left(\frac{1}{n^{\alpha+\alpha_0+1}}\right)$. Furthermore,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_4} = \frac{1}{|t_\lambda^n a'(\omega_n^j)|} + O\left(\frac{1}{n^\epsilon |t_\lambda|^n}\right) + \frac{1}{n^{\alpha+1}}\left|\frac{C_1}{a^2(\omega_n^j)}\right| + O\left(\frac{1}{n^{\alpha+\epsilon+1}}\right)$$

over Γ_4 when $n \to \infty$. The situation is similar for the segment Γ_2.

From the previous analysis of $|D_n(a-\lambda)|$ over Γ we infer by Figure 5.6 and previous analysis that, the more less values in the frontier is $\frac{1}{n^{\alpha+1}}\left|\frac{C_1}{a^2(\omega_n^j)}\right|(1+o(1))$, thus using the maximum modulus principle Corollary 2.17 that for every sufficiently large n we have

$$|\mathcal{T}_1 - \mathcal{T}_2|_\Gamma \geqslant \frac{1}{2n^{\alpha+1}}\left|\frac{C_1}{a^2(\omega_n^j)}\right|$$

and

$$\left| O\left(\left|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\right|\right) + O\left(\frac{\left|t_\lambda - \omega_n^j\right|}{n^{\alpha+1}}\right) + Q_6(n,t_\lambda)\right|_\Gamma \leqslant \frac{C}{n^{\alpha+\epsilon+1}},$$

where C is a constant. Hence, by Rouché's Theorem 2.15, $(-h_0)^{-(n+1)}D_n(a-\lambda)$ and $\mathcal{T}_1 - \mathcal{T}_2$ have the same number of zeros in E_j, that is, a unique zero. $\square$

As a consequence of Theorem 5.6, we can iterate the variable t_λ in the equation $D_n(a-\lambda) = 0$, where $D_n(a-\lambda)$ is given by (3.1). In this fashion we find the unique eigenvalue of $T_n(a)$ which is located "close" to each ω_n^j. We thus rewrite the equation $D_n(a-\lambda) = 0$ in a small neighborhood of ω_n^j as

$$\frac{1}{a'(t_\lambda)t_\lambda^{n+2}} = \frac{C_1}{a^2(t_\lambda)n^{\alpha+1}} + O\left(\frac{1}{n^{\alpha+\alpha_0+1}}\right)$$

$$\frac{1}{t_\lambda^n} = \frac{C_1 t^2 a'(t_\lambda)}{a^2(t_\lambda) n^{\alpha+1}} + O\left(\frac{1}{n^{\alpha+\alpha_0+1}}\right) = \frac{C_1 t^2 a'(t_\lambda)}{a^2(t_\lambda) n^{\alpha+1}}\left(1 + O\left(\frac{1}{n^{\alpha_0}}\right)\right).$$

Then if we reverse and take nth root above we get

$$t_{\lambda_{j,n}} = n^{\frac{\alpha+1}{n}} \omega_n^j \left(\frac{a^2(t_{\lambda_{j,n}})}{C_1 a'(t_{\lambda_{j,n}}) t^2_{\lambda_{j,n}}}\right)^{\frac{1}{n}} \cdot \big(1 + Q_7(n,j)\big)^{-\frac{1}{n}}; \tag{5.23}$$

recall C_1 from (5.19). The Theorem 5.6 insures a 1-1 correspondence between ω_n^j and $t_{\lambda_{j,n}}$, so after taking the nst root. Here the function $z^{\frac{1}{n}}$ takes its principal branch, specified by the argument in $(-\pi, \pi]$. Also notice that $Q_7(n,j) = O\left(\frac{1}{n^{\alpha_0}}\right)$ as $n \to \infty$, uniformly in $j \in (n_1, n - n_2)$, with n_1, n_2 as in Theorem 5.6.

Proof of Theorem 3.2. All the order terms in this proof work with $n \to \infty$, uniformly in $j \in (n_1, n - n_2)$, with n_1, n_2 as in Theorem 5.6.

Equation (5.23) is an implicit expression for $t_{\lambda_{j,n}}$. We manipulate it to obtain two asymptotic terms for $t_{\lambda_{j,n}}$. Remember that λ belongs to $\mathcal{D}(a)\backslash W_0$; see Figure 5.1. We can choose W so thin that $\lambda_{j,n} = a(t_{\lambda_{j,n}})$, $a'(t_{\lambda_{j,n}})$, and $t_{\lambda_{j,n}}$ are bounded and not too close to zero.

Now, if we denote $A_{j,n}$ as $\dfrac{a^2(t_{\lambda_{j,n}})}{C_1 a'(t_{\lambda_{j,n}}) t^2_{\lambda_{j,n}}}$, then

$$A_{j,n}^{\frac{1}{n}} = \exp\left(\frac{1}{n}\log A_{j,n}\right) = 1 + \frac{1}{n}\log A_{j,n} + O\left(\frac{1}{n^2}\right)$$

and $\big(1 + Q_7(n,j)\big)^{-\frac{1}{n}} = O\left(\frac{1}{n^{1+\alpha_0}}\right)$.

After expanding and multiplying the terms in parenthesis in (5.23), we obtain

$$t_{\lambda_{j,n}} = n^{\frac{\alpha+1}{n}} \omega_n^j \left(1 + \frac{1}{n}\log(A_{\lambda_{j,n}}) + Q_8(n,j)\right), \tag{5.24}$$

where $Q_8(n,j) = O\left(\frac{1}{n^{1+\alpha_0}}\right)$. Our first approximation for $t_{\lambda_{j,n}}$ is the smaller order of (5.24), that is

$$t_{\lambda_{j,n}} = n^{\frac{\alpha+1}{n}} \omega_n^j (1 + O(n^{-1})) = \omega_n^j \big(1 + Q_9(n,j)\big), \tag{5.25}$$

where $Q_9(n,j) = O\left(\frac{\log n}{n^2}\right)$, which is a consequence of $n^{\frac{\alpha+1}{n}} = \exp\left(\frac{\alpha+1}{n}\log(n)\right)$. Inserting (5.25) in (5.24), we get

$$t_{\lambda_{j,n}} = n^{\frac{\alpha+1}{n}} \omega_n^j \left[1 + \frac{1}{n} \log \left(\frac{a^2\big(\omega_n^j[1+Q_9(n,j)]\big)}{C_1 a'\big(\omega_n^j[1+Q_9(n,j)]\big)\big(\omega_n^j[1+Q_9(n,j)]\big)^2} \right) + Q_8(n,j) \right].$$

Now we use the analyticity of log, a, and a' in W, and Proposition 2.14 to obtain that

$$\log \left(\frac{a^2\big(\omega_n^j[1+Q_9(n,j)]\big)}{C_1 a'\big(\omega_n^j[1+Q_9(n,j)]\big)\big(\omega_n^j[1+Q_9(n,j)]\big)^2} \right) = \log \left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}} \right) + Q_9(n,j),$$

we can simplify the expression for $t_{\lambda_{j,n}}$ to obtain

$$t_{\lambda_{j,n}} = n^{\frac{\alpha+1}{n}} \omega_n^j \left(1 + \frac{1}{n} \log \left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}} \right) + R_2(n,j) \right),$$

where $R_2(n,j) = O\left(\frac{1}{n^{1+\alpha_0}}\right) + O\left(\frac{\log n}{n^2}\right)$. □

Proof of Theorem 3.3. All the order terms in this proof work with $n \to \infty$, uniformly in $j \in (n_1, n-n_2)$, with n_1, n_2 as in Theorem 5.6. Note that

$$n^{\frac{\alpha+1}{n}} = \exp\left(\frac{(\alpha+1)}{n} \log n \right) = 1 + \frac{(\alpha+1)}{n} \log n + O\left(\frac{\log n}{n}\right)^2. \tag{5.26}$$

Inserting (5.26) in (3.2) we obtain

$$t_{\lambda_{j,n}} = \omega_n^j \left(1 + \frac{(\alpha+1)}{n} \log n + \frac{1}{n} \log \left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}} \right) + Q_{10}(n,j) \right), \tag{5.27}$$

where $Q_{10}(n,j) = O\left(\frac{1}{n^{1+\alpha_0}}\right) + O\left(\frac{\log^2 n}{n^2}\right)$. Now we know that

$$a(z) = a(\omega_n^j) + a'(\omega_n^j)(z - \omega_n^j) + O(|z-\omega_n^j|^2), \tag{5.28}$$

applying the symbol a to (5.27) and taking $z = t_{\lambda_{j,n}}$ in (5.28), we see that,

$$\begin{aligned} a(t_\lambda) = a(\omega_n^j) + \omega_n^j a'(\omega_n^j) &\left(\frac{(\alpha+1)}{n} \log n + \frac{1}{n} \log \left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}} \right) + Q_{10}(n,j) \right) \\ &+ O\left(\frac{\log n}{n}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^{\alpha_0+1}}\right) + O\left(\frac{\log^2 n}{n^2}\right), \end{aligned}$$

the dominant order in the last equation is $Q_{10}(n,j)$, so

$$\lambda_{j,n} = a(\omega_n^j) + (\alpha+1)\omega_n^j a'(\omega_n^j) \frac{\log n}{n} + \frac{\omega_n^j a'(\omega_n^j)}{n} \log \left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}} \right) + Q_{10}(n,j)$$ □

Chapter 6

Behavior of Extreme Eigenvalues

We start this section with a technical result that enables us to invert the symbol a in a certain neighborhood of 0.

6.1 Location of eigenvalues

Lemma 6.1 (Refer [1]). *Let ρ be a small positive constant and a be the symbol in (1.2) satisfying the Properties 3.2. Then*

(i) there exist U_1, U_2 subsets of $K_\varepsilon \setminus \overline{\mathbb{D}}$ such that $a(U_1) \subseteq S_1$ and $a(U_2) \subseteq S_2$, and a restricted to $U_1 \cup U_2$ is a bijective map; moreover, for some small positive δ and each $\lambda \in S_1 \setminus R_1, S_2 \setminus R_2$, there exists a unique z_λ in U_1, U_2, respectively, such that $a(z_\lambda) = \lambda$;

(ii) for some small positive μ we have,

$$\frac{\pi}{2} - \mu < \arg(1-z) \leqslant \pi \quad \text{for every} \quad z \in U_1,$$
$$-\pi \leqslant \arg(1-z) < -\frac{\pi}{2} + \mu \quad \text{for every} \quad z \in U_2;$$

that is, the sets U_1, U_2 are located as in Figure 3.3;

(iii) z_λ is a simple zero of $a - \lambda$.

Proof. (i)

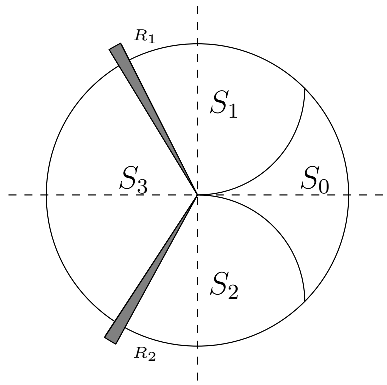


Figure 6.1: Regions S_i and R_j with $i = 1, 2, 3$ and $j = 1, 2$.

Let $U_1 \coloneqq a^{-1}(S_1)$ and $U_2 \coloneqq a^{-1}(S_2)$, see Figure 3.3. By property 1, f has an analytic continuation to K_ε, thus a has a continuous extension to $\hat{K}_\varepsilon$

Let's show the uniqueness of z_λ. Suppose that there exist z_λ and $\tilde{z}_\lambda$ in $U_1 \cup U_2$ satisfying $a(z_\lambda) = a(\tilde{z}_\lambda) = \lambda$, note that z_λ and $\tilde{z}_\lambda$ belongs to the same set U_1 or U_2, thus

$$a(z_\lambda) - a(\tilde{z}_\lambda) = 0 = \int_{\gamma_\lambda} a'(z)\,\mathrm{d}z, \tag{6.1}$$

where γ_λ is some closed polygonal curve in U_1 or U_2 from $\tilde{z}_\lambda$ to z_λ. Since f is an arbitrarily smooth function with $f(1) = 1$ and $f'(1) = 1$, we have

$$a'(z) = -\frac{\alpha}{z}(1-z)^{\alpha-1} f(z)\left(1 + \frac{1-z}{\alpha z} - \frac{(1-z)f'(z)}{\alpha f(z)}\right).$$

Now $f(z) = f'(z) = 1 + O(|1-z|)$ and $z = 1 + O(|1-z|)$ so $z^{-1}f(z) = 1 + O(|1-z|)$ then,

$$a'(z) = \, -\alpha(1-z)^{\alpha-1}(1 + O(|1-z|)) \quad (z \to 1). \tag{6.2}$$

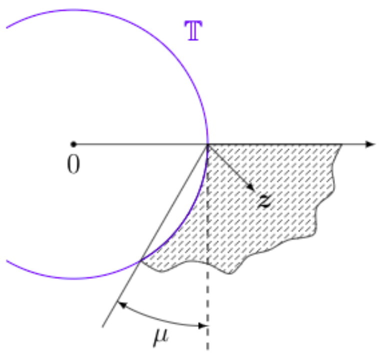
Putting together (6.1) and (6.2), as $\lambda \to 0$, we get

$$\begin{aligned} a(z_\lambda) - a(\tilde{z}_\lambda) &= -\alpha \int_{\gamma_\lambda} (1-z)^{\alpha-1}\,\mathrm{d}z + O\left(\int_{\gamma_\lambda} |1-z|^{\alpha} |\,\mathrm{d}z|\right) \\ &= (1-z_\lambda)^\alpha - (1-\tilde{z}_\lambda)^\alpha + O\left(\int_{\gamma_\lambda} |1-z|^{\alpha} |\,\mathrm{d}z|\right). \end{aligned} \tag{6.3}$$

In order to reach a contradiction, we work separately with the terms in the right of (6.3). We begin by showing that there exists a positive constant c satisfying

$$|(1-z_\lambda)^\alpha - (1-\tilde{z}_\lambda)^\alpha| \geqslant c|z_\lambda - \tilde{z}_\lambda|. \tag{6.4}$$

Suppose first that $\lambda \in S_1$. Then $z_\lambda, \tilde{z}_\lambda \in U_1$. Let I_λ be the closed line segment from $\tilde{z}_\lambda$ to z_λ. We thus have $-\frac{\pi}{2} - \mu \leqslant \arg(z-1) \leqslant 0$ for some small positive μ and every $z \in I_\lambda$ (See Figure 6.2).



Now if $z-1 = re^{in\theta}$ then $1-z = re^{in\theta+\pi}$, thus $\frac{\pi}{2} - \mu \leqslant \arg(1-z) \leqslant \pi$ which implies that $-(1-\alpha)\pi \leqslant \arg(1-z)^{\alpha-1} \leqslant (1-\alpha)\big(\mu - \frac{\pi}{2}\big)$ for every $z \in I_\lambda$,

Figure 6.2: Set U_2

similarly if $\lambda \in S_2$ we can get $(1-\alpha)(\frac{\pi}{2} - \mu) \leqslant \arg(1-z)^{\alpha-1} \leqslant \pi(1-\alpha)$. Then

$$\inf_{z\in I_\lambda} |\mathfrak{Im}(1-z)^{\alpha-1}| = \inf_{z\in I_\lambda} \big\{|1-z|^{\alpha-1}|\sin(\arg(1-z)^{\alpha-1})|\big\} \geqslant 1 \geqslant \frac{c}{\alpha} > 0$$

for some positive c. Using the parametrization $r(t) = tz_\lambda + (1-t)\tilde{z}_\lambda$ for $0 \leqslant t \leqslant 1$, we get

$$\begin{aligned}
|(1-z_\lambda)^\alpha - (1-\tilde{z}_\lambda)^\alpha| &= \alpha\left|\int_{I_\lambda} (1-z)^{\alpha-1}\,\mathrm{d}z\right| \\
&= \alpha|z_\lambda - \tilde{z}_\lambda|\left|\int_0^1 (1-tz_\lambda-(1-t)\tilde{z}_\lambda)^{\alpha-1}\,\mathrm{d}t\right| \\
&\geqslant \alpha|z_\lambda - \tilde{z}_\lambda|\mathfrak{Im}\left|\int_0^1 (1-tz_\lambda-(1-t)\tilde{z}_\lambda)^{\alpha-1}\,\mathrm{d}t\right| \\
&\geqslant \alpha|z_\lambda - \tilde{z}_\lambda|\int_0^1 \inf_{z\in I_\lambda} |\mathfrak{Im}(1-z)^{\alpha-1}|\,\mathrm{d}t \\
&\geqslant c|z_\lambda - \tilde{z}_\lambda|,
\end{aligned}$$

the third line is true since $\mathfrak{Im}(1-z) \geqslant 0$ for $\lambda \in S_1$ and $\mathfrak{Im}(1-z) \leqslant 0$ for $\lambda \in S_2$, we can change the branch of the logarithm in such a way that the function is multiplied by some vector of norm one in particular -1, where $z = t_0 z_\lambda + (t_0 - 1)z_\lambda$ with $t \in [0,1]$ which proves (6.4). On the other hand, noticing that

$$\left| \int_{\gamma_\lambda} |1-z|^\alpha \, \mathrm{d}z \right| \leqslant k_\lambda^\alpha \int_{\gamma_\lambda} |\, \mathrm{d}z| = k_\lambda^\alpha |z_\lambda - \tilde{z}_\lambda|,$$

where $k_\lambda \coloneqq \sup\{|1-z| \colon z \in \gamma_\lambda\}$ satisfies $k_\lambda \to 0$ as $\lambda \to 0$, we obtain

$$g(z) \coloneqq O\bigg(\int_{\gamma_\lambda} |1-z|^\alpha |\, \mathrm{d}z| \bigg) = o(|z_\lambda - \tilde{z}_\lambda|) \quad (\lambda \to 0), \tag{6.5}$$

because $|g(z)| \leqslant k_\lambda^\alpha |z_\lambda - \tilde{z}_\lambda|$, then $g(z) = o(|z_\lambda - \tilde{z}_\lambda|)$ by the property of k_λ. Combining the relations (6.3), (6.4), and (6.5) we obtain

$$|a(z_\lambda) - a(\tilde{z}_\lambda)| \geqslant (c - o(1))|z_\lambda - \tilde{z}_\lambda| > \frac{c}{2}|z_\lambda - \tilde{z}_\lambda|,$$

which contradicts (6.1). Note that because of the power ramification at the real positive semi-axis, a cannot be analytically extended to K_ε. We have proven that for some small positive δ and every $\lambda \in S_1 \setminus R_1, S_2 \setminus R_2$ there exists $z_\lambda \in U_1, U_2$, respectively, satisfying $a(z_\lambda) = \lambda$.

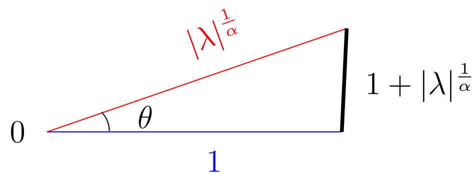
(ii) Recall that $\psi = \arg \lambda$. We know that the point z_λ is located outside of the unit disk $\mathbb{D}$, $z_\lambda \to 1$ as $\lambda \to 0$, and that

$$z_\lambda = 1 - \left(\frac{\lambda z_\lambda}{f(z_\lambda)} \right)^{\frac{1}{\alpha}},$$

which, by the smoothness of the continuation of f produces $f(z_\lambda) \to 1$ as $\lambda \to 0$. Note that $\frac{z_\lambda}{f(z_\lambda)} - 1 = o(|\lambda^{\frac{1}{\alpha}}|)$ gives us

$$z_\lambda = 1 - \lambda^{\frac{1}{\alpha}} + O(|\lambda|^{\frac{2}{\alpha}}) \quad \text{and} \quad \arg(1 - z_\lambda) = \frac{\psi}{\alpha} + O(|\lambda|^{\frac{1}{\alpha}}) \ \ (\lambda \to 0). \tag{6.6}$$

Because $\left(\frac{z_\lambda}{f(z_\lambda)}\right)^{\frac{1}{\alpha}} = 1 + O(|\lambda^{\frac{1}{\alpha}}|)$, we get $\arg(1-z_\lambda) = \arg(\lambda^{\frac{1}{\alpha}}) + \arg(1 + O(|\lambda^{\frac{1}{\alpha}}|))$, then $\arg(1 - z_\lambda) = \dfrac{\psi}{\alpha} + \arg(O(1 + |\lambda|^{\frac{1}{\alpha}}))$,



as $1+|\lambda|^{\frac{1}{\alpha}}$ is smaller, so $\tan(\theta) \sim |\lambda|^{\frac{1}{\alpha}}$
then $\theta = \arctan(\lambda^{\frac{1}{\alpha}}(1+o(1)))$,

Figure 6.3

as θ is close to 0, so arctan has a Taylor's series centered at 0, thus $\theta = O(\lambda^{\frac{1}{\alpha}})$ and this is demonstrates the second relation in (6.6).

If $\lambda \in S_1$, for a small positive μ, we must have $\frac{1}{2}\alpha\pi - \alpha\mu \leqslant \psi \leqslant \alpha\pi$. In this case, the second relation in (6.6) tells us $\frac{1}{2}\pi - \mu < \arg(1-z_\lambda) \leqslant \pi$. A similar procedure applies when $\lambda \in S_2$, we have $-\alpha\pi \leqslant \psi \leqslant -\frac{1}{2}\alpha\pi + \alpha\mu$ and thus $-\pi \leqslant \arg(1-z_\lambda) < -\frac{1}{2}\pi + \mu$. Then the sets U_1 and U_2 are located as in Figure 3.3.

(iii) Note that z_λ is a simple zero of $a - \lambda$ if and only if $a'(z_\lambda) \neq 0$. From (6.2) we get

$$a'(z_\lambda) = \frac{-\alpha}{(1-z_\lambda)^{1-\alpha}}(1+O(|1-z_\lambda|)) \quad (\lambda \to 0),$$

which combined with $z_\lambda \to 1$ as $\lambda \to 0$, gives us $\lim_{\lambda \to 0} |a'(z_\lambda)| = \infty$. □

6.2 Determinant estimation

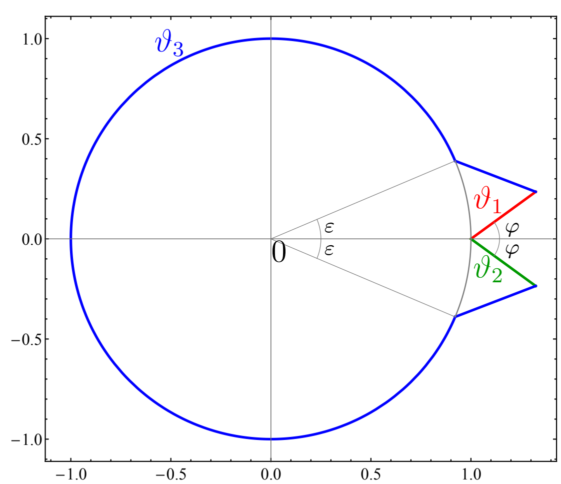


Figure 6.4: (Refer [1]) Contour ϑ

The previous proof also shows that if $\lambda \in S_0 \cup S_3$, then there is no point z_λ with $a(z_\lambda) = \lambda$. From Lemma 5.1 we know that

$$(-1)^n D_n(a-\lambda) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{t^{-(n+1)}}{(1-t)^\alpha f(t) - \lambda t}\,\mathrm{d}t = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{t^{-(n+2)}}{a(t)-\lambda}\,\mathrm{d}t \tag{6.7}$$

where $\lambda \in \mathcal{D}(a)$ and $h(0) = 1$. To deal with the Fourier integral in (6.7), we consider the contour shown in Figure 6.4. That is,

$$\begin{aligned}
\vartheta_1 &:= \{1 + x\mathrm{e}^{i\varphi} \colon 0 \leqslant x \leqslant \varepsilon\},\\
\vartheta_2 &:= \{1 + x\mathrm{e}^{-i\varphi} \colon 0 \leqslant x \leqslant \varepsilon\},\\
\vartheta_3 &:= \{x\mathrm{e}^{i\varepsilon} + (1-x)(1+\varepsilon\mathrm{e}^{i\varphi}) \colon 0 \leqslant x \leqslant 1\}\\
&\quad\ \cup \{\mathrm{e}^{i\theta} \colon \varepsilon \leqslant \theta \leqslant 2\pi - \varepsilon\}\\
&\quad\ \cup \{x(1+\varepsilon\mathrm{e}^{-i\varphi}) + (1-x)\mathrm{e}^{-i\varepsilon} \colon 0 \leqslant x \leqslant 1\},\\
\vartheta &:= \vartheta_1 \cup \vartheta_2 \cup \vartheta_3.
\end{aligned}$$

We can observe that if $\varphi \to 0$ and $\varepsilon \to 0$ then $\vartheta = \mathbb{T}$. Give ϑ the positive orientation and choose φ in the following way (see Figure 6.5):

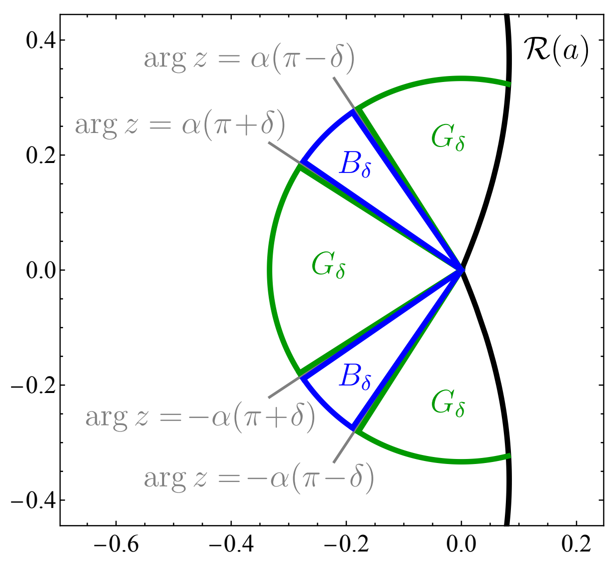
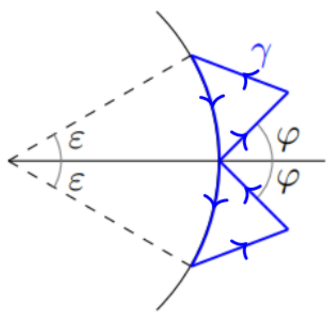


Figure 6.5: (Refer [1]) The regions in S used to determine the value of φ. If λ belongs to G_δ, B_δ we take $\varphi = 0$, $\varphi = 2\delta$, respectively.

1. Let $G_\delta \subset S$ be the set of all $\lambda \in S_1 \cup S_2$ (equivalently $z_\lambda \in U_1 \cup U_2$) with $|\arg(z_\lambda - 1)| > \delta$ (equivalently $|\psi \pm \alpha\pi| > \alpha\delta$) and all the $\lambda \in S_3$ with $|\psi \pm \alpha\pi| > \alpha\delta$ (green regions in Figure 6.5); if $\lambda \in G_\delta$ take $\varphi = 0$;

2. Let $B_\delta \subset S$ be the set $(S_1 \cup S_2 \cup S_3) \setminus G_\delta$ (blue regions in Figure 6.5); if $\lambda \in B_\delta$ take $\varphi = 2\delta$.

Let $g(z) := \dfrac{z^{-(n+2)}}{a(z) - \lambda}$. According to Lemma 6.1 for $\lambda \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$, the function g has a simple pole at z_λ.



We consider the contour γ with positive orientation, by (6.7), we have

$$(-1)^n D_n(a-\lambda) = \frac{1}{2\pi i}\int_{\mathbb{T}} g(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\vartheta} g(z)\,\mathrm{d}z - \frac{1}{2\pi i}\int_{\gamma} g(z)\,\mathrm{d}z,$$

Figure 6.6: Contour γ

and the Cauchy Residue Theorem, for every $\lambda \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$, we obtain,

$$(-1)^n D_n(a-\lambda) = -\operatorname{res}(g, z_\lambda) + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{6.8}$$

where

$$\mathcal{I}_j := \frac{1}{2\pi i}\int_{\vartheta_j} g(z)\,\mathrm{d}z \quad (j = 1, 2, 3).$$

If $\lambda \in R_1 \cup R_2 \cup S_3$ we will simply get

$$(-1)^n D_n(a-\lambda) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \tag{6.9}$$

We know that $\lambda \in \mathbb{C}$ is an eigenvalue of $T_n(a)$ if and only if $D_n(a-\lambda) = 0$, thus we are interested in the zeros of the right hand sides of (6.8) and (6.9). The following lemmas evaluate, one by one, the terms in there.

Lemma 6.2 (Refer [1]). *Suppose that* $\lambda \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$.

(i) If there exist positive constants m, M *(depending only on the symbol* a*) satisfying* $m \leqslant |\Lambda| \leqslant M$*, then*

$$\operatorname{res}(g, z_\lambda) = -\tfrac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\mathrm{e}^{\Lambda}\left(1 + O\left(\frac{|\Lambda|}{n}\right) + O\left(\frac{|\Lambda|^2}{n}\right)\right) \text{ as } n \to \infty \text{ uniformly in } \lambda.$$

(ii) $\displaystyle\lim_{|\Lambda|\to 0} \frac{\operatorname{res}(g, z_\lambda)}{\lambda^{\frac{1}{\alpha}-1}} = -\frac{1}{\alpha}$.

Proof. (i) Since $\lambda \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$, Lemma 6.1 guarantees the existence of z_λ. A direct calculation reveals that

$$\operatorname{res}(g, z_\lambda) = \lim_{z\to z_\lambda} \frac{z^{-(n+2)}(z - z_\lambda)}{a(z) - \lambda} = \lim_{z\to z_\lambda} \frac{z^{-(n+2)} - (n+2)(z - z_\lambda)z^{-(n+1)}}{a'(z)} = \frac{z_\lambda^{-(n+2)}}{a'(z_\lambda)}.$$

Now

$$\begin{aligned}
a(z) &= \frac{1}{z}(1-z)^\alpha f(z)\\
a'(z) &= -\frac{1}{z}\alpha(1-z)^{\alpha-1}f(z) + \frac{1}{z}f'(z)(1-z)^\alpha - \frac{1}{z^2}(1-z)^\alpha f(z)\\
a'(z_\lambda) &= a(z_\lambda)\left[-\frac{\alpha}{1-z_\lambda} + \frac{f'(z_\lambda)}{f(z_\lambda)} - \frac{1}{z_\lambda}\right]\\
&= \lambda\left[\frac{-\alpha f(z_\lambda)z_\lambda + f'(z_\lambda)(1-z_\lambda)z_\lambda - (1-z_\lambda)f(z_\lambda)}{(1-z_\lambda)z_\lambda f(z_\lambda)}\right].
\end{aligned}$$

Then $\displaystyle\operatorname{res}(g, z_\lambda) = \frac{z_\lambda^{-(n+1)}(z_\lambda - 1)f(z_\lambda)}{\lambda((\alpha-1)z_\lambda f(z_\lambda) + f(z_\lambda) + z_\lambda(z_\lambda - 1)f'(z_\lambda))}$.

Using the equation (6.6) and the smoothness of the continuation of f in K_ε, we get

$$f(z_\lambda) = 1 + O(|\lambda|^{\frac{1}{\alpha}}) \quad \text{and} \quad f'(z_\lambda) = f'(1) + O(|\lambda|^{\frac{1}{\alpha}}),$$

which combined with $\log(1-z) = -z + O(|z|^2)$ $(z \to 0)$ gives us

$$\begin{aligned}
\operatorname{res}(g, z_\lambda) &= \frac{-\lambda^{\frac{1}{\alpha}-1}\left[\exp(-(n+1)\log(1 - \lambda^{\frac{1}{\alpha}} + O(|\lambda^{\frac{2}{\alpha}}|)))\right](1 + O(|\lambda|^{\frac{1}{\alpha}}))}{(\alpha-1)(1 + O(|\lambda|^{\frac{1}{\alpha}})) + 1 + O(|\lambda|^{\frac{1}{\alpha}}) - \lambda^{\frac{1}{\alpha}}f'(1)(1 + O(|\lambda|^{\frac{1}{\alpha}}))}\\
&= \frac{-\lambda^{\frac{1}{\alpha}-1}\left[\exp(-(n+1)\log(1 - \lambda^{\frac{1}{\alpha}} + O(|\lambda^{\frac{2}{\alpha}}|)))\right](1 + O(|\lambda|^{\frac{1}{\alpha}}))}{\alpha - \lambda^{\frac{1}{\alpha}}f'(1)}.
\end{aligned}$$

Now $\log(1-\lambda^{\frac{1}{\alpha}}+O(|\lambda^{\frac{2}{\alpha}}|)) = -\lambda^{\frac{1}{\alpha}}+O(|\lambda^{\frac{2}{\alpha}}|)$ and $\alpha-\lambda^{\frac{1}{\alpha}}f'(1) = \alpha(1+O(|\lambda^{\frac{1}{\alpha}}|))$ so

$$\exp(-(n+1)\log(1-\lambda^{\frac{1}{\alpha}}+O(|\lambda^{\frac{2}{\alpha}}|))) = \exp(\Lambda)\exp(O(n|\lambda^{\frac{2}{\alpha}}|)).$$

Then

$$\operatorname{res}(g,z_\lambda) = -\frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\mathrm{e}^{\Lambda}(1+O(|\lambda|^{\frac{1}{\alpha}}))(1+O(n|\lambda|^{\frac{2}{\alpha}})) = -\frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\mathrm{e}^{\Lambda}(1+O(n|\lambda|^{\frac{1}{\alpha}})+O(n|\lambda|^{\frac{2}{\alpha}})).$$

Finally, recalling Λ from (3.5) we obtain the first part of the lemma. The limit in (ii) can be calculated directly. $\square$

Let $\hat{\vartheta}_1 := \log\vartheta_1$. Thus $\hat{\vartheta}_1$ is a path from 0 to $\log(1+\varepsilon\mathrm{e}^{i\varphi}) = \hat{\varepsilon}\mathrm{e}^{i\hat{\varphi}}$ with $\hat{\varepsilon}$ and $\hat{\varphi}$ satisfying

$$\hat{\varepsilon} = \varepsilon + O(\varepsilon^2) \quad \text{and} \quad \hat{\varphi} = \varphi + O(\varepsilon).$$

Analogously, let $\hat{\vartheta}_2 := \log\vartheta_2$. Thus $\hat{\vartheta}_2$ is a path from $\log(1+\varepsilon\mathrm{e}^{-i\hat{\varphi}}) = \hat{\varepsilon}\mathrm{e}^{-i\hat{\varphi}}$ to 0. For $-\pi<\beta\leqslant\pi$ let $\infty\mathrm{e}^{i\beta}$ be $\lim\limits_{s\to\infty} s\mathrm{e}^{i\beta}$. The following lemma is the heart of the calculation. It gives us asymptotic expansions for $\mathcal{I}_1$ and $\mathcal{I}_2$ with the disadvantage of handling complex integration paths.

Lemma 6.3 (Refer [1])**.** *Suppose that $\lambda\in S_1\cup S_2\cup S_3$.*

(i) If there exist positive constants m and M (depending only on the symbol a) satisfying $m\leqslant|\Lambda|\leqslant M$, then

$$\mathcal{I}_1 = \frac{|\Lambda|^{1-\alpha}}{2\pi i(n+1)^{1-\alpha}}\left(\int_0^{\infty\mathrm{e}^{i\hat{\varphi}}}\frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{-i\alpha\pi}v^{\alpha}-\mathrm{e}^{i\psi}}\,\mathrm{d}v + O\Big(\frac{1}{n}\Big)\right),$$
$$\mathcal{I}_2 = -\frac{|\Lambda|^{1-\alpha}}{2\pi i(n+1)^{1-\alpha}}\left(\int_0^{\infty\mathrm{e}^{-i\hat{\varphi}}}\frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{i\alpha\pi}v^{\alpha}-\mathrm{e}^{i\psi}}\,\mathrm{d}v + O\Big(\frac{1}{n}\Big)\right).$$

(ii) If $|\Lambda|\to 0$, then

$$\mathcal{I}_1 \sim \frac{\mathrm{e}^{i\alpha\pi}\Gamma(1-\alpha)}{2\pi i(n+1)^{1-\alpha}} \quad \textit{and} \quad \mathcal{I}_2 \sim \frac{\mathrm{e}^{i\alpha\pi}\Gamma(1-\alpha)}{2\pi i(n+1)^{1-\alpha}}.$$

Where all the asymptotic relations work with $n\to\infty$ uniformly in λ.

Proof. All the order terms in this proof work with $n \to \infty$ and $\lambda \to 0$. Consider first the integral $\mathcal{I}_1$ and make the variable change $v = \mathrm{e}^u$. Then

$$2\pi i\,\mathcal{I}_1 = \int_{\hat{\vartheta}_1} \frac{\mathrm{e}^{-(n+1)u}}{a(\mathrm{e}^u) - \lambda}\,\mathrm{d}u. \tag{6.10}$$

We can write

$$a(\mathrm{e}^u) = \mathrm{e}^{-u}(1-\mathrm{e}^u)^\alpha f(\mathrm{e}^u) = (-u)^\alpha \hat{f}(u),$$

where $\hat{f}(u) = \frac{f(\mathrm{e}^u)}{\mathrm{e}^u}\Big(1+\frac{u}{2}+\frac{u^2}{6}+\cdots\Big)^\alpha$ which, by property 1, belongs to $C^2(\hat{\vartheta}_1)$. Note that $\hat{f}(0) = f(1) = 1$ and that $(-u)^\alpha$ equals $\mathrm{e}^{-i\alpha\pi}u^\alpha$ when $u \in \hat{\vartheta}_1$ and $\mathrm{e}^{i\alpha\pi}u^\alpha$ when $u \in \hat{\vartheta}_2$. Using the function

$$k(u,\lambda) := \frac{1}{(-u)^\alpha \hat{f}(u) - \lambda} - \frac{1}{(-u)^\alpha - \lambda}$$

we split $\mathcal{I}_1$ as

$$2\pi i\,\mathcal{I}_1 = \mathcal{I}_{1,1} + \mathcal{I}_{1,2}, \tag{6.11}$$

where

$$\mathcal{I}_{1,1} := \int_{\hat{\vartheta}_1} \frac{\mathrm{e}^{-(n+1)u}}{(-u)^\alpha - \lambda}\,\mathrm{d}u \quad \text{and} \quad \mathcal{I}_{1,2} := \int_{\hat{\vartheta}_1} k(u,\lambda)\mathrm{e}^{-(n+1)u}\,\mathrm{d}u.$$

As we will see, in norm, the integral $\mathcal{I}_{1,2}$ is much smaller than $\mathcal{I}_{1,1}$. Thus we need to estimate $\mathcal{I}_{1,2}$ and we will do it by finding a uniform bound for $|k|$. To this end, note that $\hat{f}(u) = 1 + O(u)$ $(u \to 0)$ and consider another variable change: $u = |\lambda|^{\frac{1}{\alpha}}v$, remember that $\frac{\lambda}{|\lambda|} = \mathrm{e}^{i\psi}$, thus $(-u)^\alpha = |\lambda|(-v)^\alpha$ and

$$\begin{aligned}\frac{1}{(-u)^\alpha \hat{f}(u) - \lambda} &= \frac{1}{|\lambda|}\frac{1}{(-v)^\alpha + O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1}) - \mathrm{e}^{i\psi}}\\ \frac{1}{(-u)^\alpha - \lambda} &= \frac{1}{|\lambda|}\frac{1}{(-v)^\alpha - \mathrm{e}^{i\psi}}.\end{aligned}$$

We get

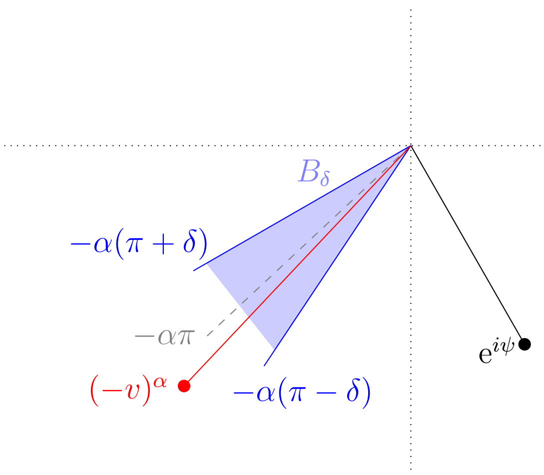
$$k(|\lambda|^{\frac{1}{\alpha}}v,\lambda) = \frac{O(|\lambda|^{\frac{1}{\alpha}-1}|v|^{\alpha+1})}{((-v)^\alpha - \mathrm{e}^{i\psi})((-v)^\alpha - \mathrm{e}^{i\psi} + O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1}))}.$$

The path $\hat{\vartheta}_1$ is close to the line segment given by $\{x\mathrm{e}^{i\hat{\varphi}} \colon 0 \leqslant x \leqslant \hat{\varepsilon}\}$. Thus for $u \in \hat{\vartheta}_1$ we have $\arg(-u)^\alpha = \arg(-v)^\alpha \sim \alpha(\hat{\varphi} - \pi)$ and we are ready to show that the denominator of $|k|$ is bounded away from 0.

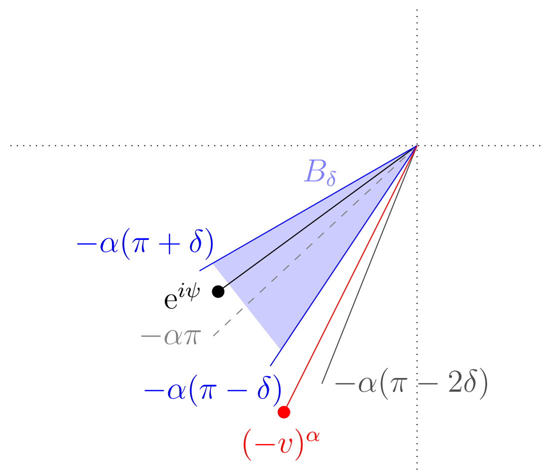
Suppose that $\lambda \in G_\delta$ (see Figure 6.5). Then $\hat{\varphi} = 0$, $(-v)^\alpha$ lies arbitrarily close to the ray with argument $-\alpha\pi$, and $\mathrm{e}^{i\psi}$ lies on $\mathbb{T}$ with $|\psi - \alpha\pi| > \alpha\delta$, (see Figure 6.7a) giving us

$$|(-v)^\alpha - \mathrm{e}^{i\psi}| \geqslant \alpha\delta > \frac{\alpha\delta}{2}. \tag{6.12}$$

If $\lambda \in B_\delta$ (see Figure 6.5), then $\hat{\varphi} = 2\delta$, $(-v)^\alpha$ lies arbitrarily close to the ray with argument $\alpha(2\delta - \pi)$, and $\mathrm{e}^{i\psi}$ lies on $\mathbb{T}$ with $|\psi - \alpha\pi| \leqslant \alpha\delta$, giving us (6.12) again (see Figure 6.7b).



(a) Case $\lambda \in G_\delta$



(b) Case $\lambda \in B_\delta$

Figure 6.7

For the second factor in the denominator of $|k|$, note that $|(-v)^\alpha - \mathrm{e}^{i\psi} + O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1})|$ attains its minimum value when $|v| \sim 1$ and thus the order term will be bounded by $|\lambda|^{\frac{1}{\alpha}} < \rho^{\frac{1}{\alpha}}$, which can be taken arbitrarily small. Then we get

$$|(-v)^\alpha - \mathrm{e}^{i\psi} + O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1})| > \frac{\alpha\delta}{4}. \tag{6.13}$$

Using (6.12) and (6.13) we get the bound $|k(|\lambda|^{\frac{1}{\alpha}}v, \lambda)| \leqslant c_2|\lambda|^{\frac{1}{\alpha}-1}|v|^{\alpha+1}$ (or equivalently $|k(u,\lambda)| \leqslant c_2|\lambda|^{-2}|u|^{\alpha+1}$) where c_2 is a positive constant not depending on λ or v.

Thus,

$$\begin{aligned}
|\mathcal{I}_{1,2}| &\leqslant \int_{\hat{\vartheta}_1} |k(u,\lambda)\mathrm{e}^{-(n+1)u}||\,\mathrm{d}u| \\
&\leqslant \frac{c_2}{|\lambda|^2}\int_{\hat{\vartheta}_1} |u|^{\alpha+1}|\mathrm{e}^{-(n+1)u}||\,\mathrm{d}u| \\
&= \frac{c_2}{(n+1)^{\alpha+2}|\lambda|^2}\int_0^{\hat{\varepsilon}\mathrm{e}^{i\hat{\varphi}}} |w|^{\alpha+1}|\mathrm{e}^{-w}||\,\mathrm{d}w| \\
&\leqslant \frac{c_2}{(n+1)^{\alpha+2}|\lambda|^2}\int_0^{\infty\mathrm{e}^{i\hat{\varphi}}} |w|^{\alpha+1}|\mathrm{e}^{-w}||\,\mathrm{d}w|,
\end{aligned}$$

where in the third line we shifted to the variable $w=(n+1)u$. Now we consider the following contour,

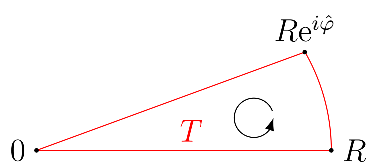


Figure 6.8: Contour T

In T the function $|w|^{\alpha+1}|\mathrm{e}^{-w}|$ has not singularities and it is bounded, note that $|w|^{\alpha+1}|\mathrm{e}^{-w}|=o(1)$ when $|w|\to\infty$. Using the dominated convergence Theorem 2.4, we have that $\lim\limits_{R\to\infty}\int_R^{R\mathrm{e}^{i\hat{\varphi}}} |w|^{\alpha+1}|\mathrm{e}^{-w}||\mathrm{d}w|=0$, thus

$$\begin{aligned}
|\mathcal{I}_{1,2}| &\leqslant \frac{c_2}{(n+1)^{\alpha+2}|\lambda|^2}\int_0^{\infty} w^{\alpha+1}\mathrm{e}^{-w}\,\mathrm{d}w \\
&= \frac{c_2\,\Gamma(\alpha+2)}{(n+1)^{\alpha+2}|\lambda|^2}.
\end{aligned}$$

The previous calculation gives us

$$\mathcal{I}_{1,2} = O\left(\frac{1}{n^{\alpha+2}|\lambda|^2}\right) = O\left(\frac{1}{n^{2-\alpha}|\Lambda|^{2\alpha}}\right) \tag{6.14}$$

uniformly in λ. Now we work with $\mathcal{I}_{1,1}$. Write

$$\mathcal{I}_{1,1} = \mathcal{I}_{1,1,1} - \mathcal{I}_{1,1,2}, \tag{6.15}$$

where

$$\mathcal{I}_{1,1,1} := \int_0^{\infty\mathrm{e}^{i\hat{\varphi}}} \frac{\mathrm{e}^{-(n+1)u}}{(-u)^\alpha-\lambda}\,\mathrm{d}u \quad \text{and} \quad \mathcal{I}_{1,1,2} := \int_{\hat{\varepsilon}\mathrm{e}^{i\hat{\varphi}}}^{\infty\mathrm{e}^{i\hat{\varphi}}} \frac{\mathrm{e}^{-(n+1)u}}{(-u)^\alpha-\lambda}\,\mathrm{d}u.$$

For $\mathcal{I}_{1,1,2}$ consider the change of variable $w=u\mathrm{e}^{-i\hat{\varphi}}$. Thus

$$|\mathcal{I}_{1,1,2}| = \left|\int_{\hat{\varepsilon}}^{\infty} \frac{\mathrm{e}^{i\hat{\varphi}}\mathrm{e}^{-(n+1)\mathrm{e}^{i\hat{\varphi}}w}}{(-w\mathrm{e}^{i\hat{\varphi}})^\alpha-\lambda}\,\mathrm{d}w\right|,$$

note that $|(-we^{i\hat{\varphi}})^\alpha - \lambda| \geqslant ||w|^\alpha - |\lambda|| > |\hat{\varepsilon}^\alpha - |\lambda||$ hence,

$$|\mathcal{I}_{1,1,2}| \leqslant \frac{1}{|\hat{\varepsilon}^\alpha - |\lambda||} \int_{\hat{\varepsilon}}^{\infty} \mathrm{e}^{-(n+1)w\cos\hat{\varphi}} \, \mathrm{d}w$$
$$= \frac{\mathrm{e}^{-(n+1)\hat{\varepsilon}\cos\hat{\varphi}}}{(n+1)|\hat{\varepsilon}^\alpha - |\lambda|| \cos\hat{\varphi}}. \tag{6.16}$$

Since $\hat{\varphi}$ is a small non-negative constant we get $-\cos\hat{\varphi} < -\dfrac{1}{2}$ and $|\lambda| < \rho$, which can be chosen satisfying $\rho < \hat{\varepsilon}^\alpha$, equation (6.16) shows that

$$\mathcal{I}_{1,1,2} = O\Big(\frac{\mathrm{e}^{-\frac{1}{2}\hat{\varepsilon}n}}{n}\Big) \text{ uniformly in } \lambda.$$

Taking again the variable change $u = |\lambda|^{\frac{1}{\alpha}}v$, and putting together (6.11), (6.14), (6.15), and (6.16) we obtain

$$\mathcal{I}_1 = \frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i}\bigg(\int_0^{\infty \mathrm{e}^{i\hat{\varphi}}} \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{-i\alpha\pi}v^\alpha - \mathrm{e}^{i\psi}} \, \mathrm{d}v + O\Big(\frac{1}{n|\Lambda|^{\alpha+1}}\Big)\bigg) \tag{6.17}$$

uniformly in λ. A result for $\mathcal{I}_2$ can be obtained readily by changing every $\hat{\varphi}$ by $-\hat{\varphi}$, getting

$$\mathcal{I}_2 = -\frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i}\bigg(\int_0^{\infty \mathrm{e}^{-i\hat{\varphi}}} \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{i\alpha\pi}v^\alpha - \mathrm{e}^{i\psi}} \, \mathrm{d}v + O\Big(\frac{1}{n|\Lambda|^{\alpha+1}}\Big)\bigg) \tag{6.18}$$

uniformly in λ. For proving (i) suppose that $m \leqslant |\Lambda| \leqslant M$. Then the result is immediate from (6.17) and (6.18).

For proving (ii) take $|\Lambda| \to 0$ and assume first that $\lambda \in G_\delta$ (see Figure 6.5), thus $\varphi = \hat{\varphi} = 0$. From equation (6.10), with

$$\hat{f}(u) = \hat{f}(0) + \hat{f}'(0)O(u) = 1 + O(u) \quad (u \to 0)$$

and the change of variable $u = |\lambda|^{\frac{1}{\alpha}}v$, we get

$$2\pi i\,\mathcal{I}_1 = \int_{\vartheta_1} \frac{\mathrm{e}^{-(n+1)u}}{\mathrm{e}^{-i\alpha\pi}u^\alpha \hat{f}(u) - \lambda} \, \mathrm{d}u$$
$$= |\lambda|^{\frac{1}{\alpha}-1} \int_{\vartheta_1'} \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{-i\alpha\pi}v^\alpha \hat{f}(|\lambda|^{\frac{1}{\alpha}}v) - \mathrm{e}^{i\psi}} \, \mathrm{d}v$$
$$= |\lambda|^{\frac{1}{\alpha}-1} \int_{\vartheta_1'} \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{-i\alpha\pi}v^\alpha - \mathrm{e}^{i\psi} + O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1})} \, \mathrm{d}v$$
$$= |\lambda|^{\frac{1}{\alpha}-1}(\mathcal{J}_{1,1} + \mathcal{J}_{1,2}), \tag{6.19}$$

where ϑ_1' is a continuous path in $\mathbb{C}$ starting at 0 and ending at $\hat{\varepsilon}|\lambda|^{-\frac{1}{\alpha}}$, and

$$\begin{aligned}\mathcal{J}_{1,1} &\coloneqq \int_{\vartheta_{1,1}'} \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{-i\alpha\pi}v^{\alpha} - \mathrm{e}^{i\psi} + O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1})}\,\mathrm{d}v,\\ \mathcal{J}_{1,2} &\coloneqq \int_{\vartheta_{1,2}'} \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{-i\alpha\pi}v^{\alpha} - \mathrm{e}^{i\psi} + O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1})}\,\mathrm{d}v;\end{aligned} \tag{6.20}$$

here $\vartheta_{1,1}'$ and $\vartheta_{1,2}'$ are the portions of ϑ_1' from 0 to 1 and from 1 to $\hat{\varepsilon}|\lambda|^{-\frac{1}{\alpha}}$, respectively. We proceed to find order bounds for $\mathcal{J}_{1,1}$ and $\mathcal{J}_{1,2}$. The former will be easy but the latter will require a lot more work.

Consider the integral $\mathcal{J}_{1,1}$. The term $O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1}) = O(|\lambda|^{\frac{1}{\alpha}})$ is arbitrarily small and the denominator in the integrand of $\mathcal{J}_{1,1}$ in (6.20) has a zero at some point close to $v = \mathrm{e}^{i(\alpha\pi+\psi)}$. For $\lambda \in G_\delta$ we have $|\alpha\pi - \psi| > \alpha\delta$, thus

$$\begin{aligned}|\mathrm{e}^{-i\alpha\pi}v^{\alpha} - \mathrm{e}^{i\psi} + O(|\lambda|^{\frac{1}{\alpha}})| &\geqslant |\mathrm{e}^{-i\alpha\pi}v^{\alpha} - \mathrm{e}^{i\psi}| - O(|\lambda|^{\frac{1}{\alpha}})\\ &\geqslant |v^{\alpha}||i(\psi - \alpha\pi) - O((\psi-\alpha\pi)^2)| + O(|\lambda|^{\frac{1}{\alpha}})\\ &\geqslant |\psi - \alpha\pi| + O(|\lambda|^{\frac{1}{\alpha}})\\ &> \alpha\delta + O(|\lambda|^{\frac{1}{\alpha}})\\ &> \alpha\delta.\end{aligned} \tag{6.21}$$

We thus have

$$|\mathcal{J}_{1,1}| \leqslant \frac{1}{\alpha\delta}\int_{\vartheta_{1,1}'} \mathrm{e}^{-|\Lambda|v}\,\mathrm{d}v \leqslant \frac{1}{\alpha\delta}. \tag{6.22}$$

To find an order bound for $\mathcal{J}_{1,2}$ we will go through three steps: In the first one, we split it as $\mathcal{J}_{1,2,1} + \mathcal{J}_{1,2,2}$, in the second step we bound $\mathcal{J}_{1,2,1}$, and in the third step we study $\mathcal{J}_{1,2,2}$ for the cases $0 < \alpha < \frac{1}{2}$, $\alpha = \frac{1}{2}$, and $\frac{1}{2} < \alpha < 1$ separately. Finally we will put all together.

Step 1: Consider the function

$$\ell(v,\lambda) \coloneqq \frac{1}{\mathrm{e}^{-i\alpha\pi}v^{\alpha} - \mathrm{e}^{i\psi} + O(|\lambda|^{\frac{1}{\alpha}}|v|^{\alpha+1})} - \frac{1}{\mathrm{e}^{-i\alpha\pi}v^{\alpha}}$$

and split $\mathcal{J}_{1,2}$ as

$$\mathcal{J}_{1,2} = \mathcal{J}_{1,2,1} + \mathcal{J}_{1,2,2} \tag{6.23}$$

where

$$\mathcal{J}_{1,2,1} := \int_{\vartheta'_{1,2}} \frac{\mathrm{e}^{-|\Lambda| v}}{\mathrm{e}^{-i\alpha\pi} v^{\alpha}} \,\mathrm{d}v \quad \text{and} \quad \mathcal{J}_{1,2,2} := \int_{\vartheta'_{1,2}} \ell(v,\lambda) \mathrm{e}^{-|\Lambda| v} \,\mathrm{d}v.$$

Step 2: Considering the variable change $w = |\Lambda| v$ we get

$$|\mathcal{J}_{1,2,1}| \leqslant$$

$$\begin{aligned} |\mathcal{J}_{1,2,1}| &= \left| \frac{\mathrm{e}^{i\alpha\pi}}{|\Lambda|^{1-\alpha}} \int_{|\Lambda|}^{\hat{\varepsilon}(n+1)} w^{-\alpha} \mathrm{e}^{-w} \,\mathrm{d}w \right| \\ &\leqslant \frac{1}{|\Lambda|^{1-\alpha}} \int_{|\Lambda|}^{\hat{\varepsilon}(n+1)} |w^{-\alpha}| |\mathrm{e}^{-w}| |\,\mathrm{d}w| \\ &\leqslant \frac{1}{|\Lambda|^{1-\alpha}} \int_{0}^{\infty} w^{-\alpha} \mathrm{e}^{-w} \,\mathrm{d}w \\ &= \frac{\Gamma(1-\alpha)}{|\Lambda|^{1-\alpha}}. \end{aligned} \tag{6.24}$$

Step 3: Using (6.21), there exists a constant c_1 satisfying

$$|\ell(v,\lambda)| \leqslant \frac{1 + O(|\lambda|^{\frac{1}{\alpha}} |v|^{\alpha+1})}{|v|^{2\alpha} |1 - v^{-\alpha} \mathrm{e}^{i(\psi+\alpha\pi)} + O(|\lambda|^{\frac{1}{\alpha}} |v|)|} \leqslant \frac{1 + c_1 |\lambda|^{\frac{1}{\alpha}} |v|^{\alpha+1}}{\alpha\delta |v|^{2\alpha}},$$

so that, for every $v \in \vartheta'_{1,2}$, we have

$$|\ell(v,\lambda)| \leqslant \frac{1}{\alpha\delta |v|^{2\alpha}} + c_1 |\lambda|^{\frac{1}{\alpha}} |v|^{1-\alpha}.$$

Then using the variable change $w = |\Lambda| v$ again, we get

$$\begin{aligned} |\mathcal{J}_{1,2,2}| &\leqslant \frac{1}{\alpha\delta} \int_{\vartheta'_{1,2}} \frac{\mathrm{e}^{-|\Lambda| v}}{|v|^{2\alpha}} |\,\mathrm{d}v| + c_1 |\lambda|^{\frac{1}{\alpha}} \int_{\vartheta'_{1,2}} |v|^{1-\alpha} \mathrm{e}^{-|\Lambda| v} |\,\mathrm{d}v| \\ &\leqslant \frac{|\Lambda|^{2\alpha-1}}{\alpha\delta} \int_{|\Lambda|}^{\hat{\varepsilon}(n+1)} w^{-2\alpha} \mathrm{e}^{-w} \,\mathrm{d}w + \frac{c_1 |\Lambda|^{\alpha-1}}{n+1} \int_{|\Lambda|}^{\hat{\varepsilon}(n+1)} w^{1-\alpha} \mathrm{e}^{-w} \,\mathrm{d}w \\ &\leqslant \frac{|\Lambda|^{2\alpha-1}}{\alpha\delta} \int_{|\Lambda|}^{\hat{\varepsilon}(n+1)} w^{-2\alpha} \mathrm{e}^{-w} \,\mathrm{d}w + \frac{c_1 \Gamma(2-\alpha)}{(n+1)|\Lambda|^{1-\alpha}} \\ &= \frac{1}{|\Lambda|^{1-\alpha}} \left(\frac{|\Lambda|^{\alpha}}{\alpha\delta} \hat{\mathcal{J}} + \frac{c_1 \Gamma(2-\alpha)}{n+1} \right), \end{aligned} \tag{6.25}$$

where

$$\hat{\mathcal{J}} := \int_{|\Lambda|}^{\hat{\varepsilon}(n+1)} w^{-2\alpha} \mathrm{e}^{-w} \,\mathrm{d}w \leqslant \int_{|\Lambda|}^{\infty} w^{-2\alpha} \mathrm{e}^{-w} \,\mathrm{d}w.$$

The integral $\hat{\mathcal{J}}$ can be estimated as follows. Suppose that $\alpha = \frac{1}{2}$, integrating by parts we obtain

$$\hat{\mathcal{J}} \leqslant -\frac{\ln|\Lambda|}{\mathrm{e}^{|\Lambda|}} + \int_{|\Lambda|}^{\infty} \mathrm{e}^{-w} \ln w \, \mathrm{d}w, \tag{6.26}$$

which means that $\hat{\mathcal{J}} = O(\ln|\Lambda|)$ $(|\Lambda| \to 0)$ because $\ln w = O(\mathrm{e}^{-w})$ when $(w \to \infty)$. Suppose that $0 < \alpha < \frac{1}{2}$, since $0 < 1 - 2\alpha < 1$ in this case, we obtain

$$\hat{\mathcal{J}} \leqslant \int_{0}^{\infty} w^{-2\alpha} \mathrm{e}^{-w} \, \mathrm{d}w = \Gamma(1-2\alpha) \tag{6.27}$$

which means that $\hat{\mathcal{J}} = O(1)$ $(|\Lambda| \to 0)$. Finally, suppose that $\frac{1}{2} < \alpha < 1$, integrating by parts we get

$$\hat{\mathcal{J}} \leqslant \frac{\mathrm{e}^{-|\Lambda|}}{(2\alpha-1)|\Lambda|^{2\alpha-1}} - \frac{1}{1-2\alpha} \int_{|\Lambda|}^{\infty} w^{1-2\alpha} \mathrm{e}^{-w} \, \mathrm{d}w, \tag{6.28}$$

which means that $\hat{\mathcal{J}} = O\left(\frac{1}{|\Lambda|^{2\alpha-1}}\right)$ $(|\Lambda| \to 0)$, because the integral in the second term is bigger them $\Gamma(2\alpha)$. Remember that $m \leqslant |\Lambda| \leqslant M$, using (6.26), (6.27), and (6.28) in (6.25) we obtain

$$\mathcal{J}_{1,2,2} = o\left(\frac{1}{|\Lambda|^{1-\alpha}}\right) \quad (|\Lambda| \to 0). \tag{6.29}$$

Putting together (6.23), (6.24), and (6.29) we get

$$\mathcal{J}_{1,2} = \frac{\mathrm{e}^{i\alpha\pi}\Gamma(1-\alpha)}{|\Lambda|^{1-\alpha}}(1+o(1)),$$

which combined with (6.19) and (6.22) gives us

$$\begin{aligned}
\mathcal{I}_1 &= \frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i}\left(O(1) + \frac{\mathrm{e}^{i\alpha\pi}\Gamma(1-\alpha)}{|\Lambda|^{1-\alpha}}(1+o(1))\right) \\
&= \frac{|\lambda|^{\frac{1}{\alpha}-1}\mathrm{e}^{i\alpha\pi}\Gamma(1-\alpha)}{2\pi i|\Lambda|^{1-\alpha}}(O(|\Lambda|^{\alpha-1}) + 1 + o(1)) \\
&= \frac{\mathrm{e}^{i\alpha\pi}\Gamma(1-\alpha)}{2\pi i(n+1)^{1-\alpha}}(1+o(1)),
\end{aligned}$$

where in the third line we changed $O(|\Lambda|^{\alpha-1})$ for $o(1)$ because Λ is bounded, which shows the assertion (ii) for the case $\lambda \in G_\delta$. The case $\lambda \in B_\delta$ can be readily obtained as well as the corresponding result for $\mathcal{I}_2$. □

The result in Lemma 6.3 is a neat asymptotic approach but we want to rotate the integration path to the real axis. Recall that $\psi = \arg\lambda$. To this end, choose a small positive δ and consider the following subsets of S:

$$R_1 \coloneqq \{\lambda \in S \colon \alpha(\pi - \delta) \leqslant \psi < \alpha\pi\};$$
$$R_2 \coloneqq \{\lambda \in S \colon -\alpha\pi < \psi \leqslant -\alpha(\pi - \delta)\}.$$

We thus know that if $\lambda \in R_1 \cup R_2$, then $\varphi = 2\delta$, and if $\lambda \in G_\delta$, then $\varphi = 0$. Let χ_A stand for the characteristic function of the set A. Depending on the choice of φ we can rotate the integrals in Lemma 6.3 obtaining the following two lemmas.

Lemma 6.4 (Refer [1]). *Suppose that there exists constants m, M (depending only on the symbol a) satisfying $m \leqslant |\Lambda| \leqslant M$. For $\lambda \in S_1 \cup S_2 \cup S_3$ with $\lambda \to 0$ as $n \to \infty$, we have*

$$\mathcal{I}_1 = \frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\mathrm{e}^{\Lambda}\chi_{R_2}(\lambda) + \frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i}\int_0^\infty \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{-i\alpha\pi}v^\alpha - \mathrm{e}^{i\psi}}\,\mathrm{d}v + O\left(\frac{1}{n^{2-\alpha}}\right),$$
$$\mathcal{I}_2 = \frac{-1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\mathrm{e}^{\Lambda}\chi_{R_1}(\lambda) - \frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i}\int_0^\infty \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{i\alpha\pi}v^\alpha - \mathrm{e}^{i\psi}}\,\mathrm{d}v + O\left(\frac{1}{n^{2-\alpha}}\right).$$

Proof. In this proof, all the order terms work with $n \to \infty$ uniformly in λ. Let $\lambda \in G_\delta$ (see Figure 6.5). Then $\hat{\varphi} = 0$ and the result follows directly from Lemma 6.3 part (i), since for (6.17) (6.18) we have

$$\frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i}O\left(\frac{1}{n|\Lambda|^{\alpha+1}}\right) = O\left(\frac{1}{n^{2-\alpha}|\Lambda|^{2\alpha}}\right) = O\left(\frac{1}{n^{2-\alpha}}\right).$$

Consider now the case $\lambda \in R_1 \cup R_2$ (see B_δ in Figure 6.5), then $\varphi = 2\delta$. In order to rotate our integration path, for a large positive R, we consider the positively orientated triangle T with vertices 0, R, and $R\mathrm{e}^{i\hat{\varphi}}$ (see Figure 6.8). Let h be the function

$$h(v) \coloneqq \frac{\mathrm{e}^{-|\Lambda|v}}{\mathrm{e}^{-i\alpha\pi}v^\alpha - \mathrm{e}^{i\psi}};$$

thus equation (6.17) can be written as

$$\mathcal{I}_1 = \frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i}\int_0^{\infty\mathrm{e}^{i\hat{\varphi}}} h(v)\,\mathrm{d}v + O\left(\frac{1}{n^{2-\alpha}|\Lambda|^{2\alpha}}\right).$$

The function h has a singularity at $v_0 = \mathrm{e}^{i(\pi+\frac{\psi}{\alpha})}$. Assume that $\lambda \in R_2$. Then v_0 is enclosed by T because, in this case, we must have $0 < \arg v_0 \leqslant \delta < \hat{\varphi}$, since

$$\int_T h(v)\,\mathrm{d}v = \int_0^R h(v)\,\mathrm{d}v + \int_R^{Re^{i\hat{\varphi}}} h(v)\,\mathrm{d}v - \int_0^{re^{i\hat{\varphi}}} h(v)\,\mathrm{d}v = \operatorname{res}(h, v_0).$$

Note that $h(v)$ is bounded and $h(v) = o(1)$ when $|v| \to \infty$, using the dominated convergence Theorem 2.4, we have that $\lim_{R\to\infty} \int_R^{Re^{i\hat{\varphi}}} h(v)\,\mathrm{d}v = 0$, as $\Lambda = (n+1)\lambda^{\frac{1}{\alpha}}$ then

$$\begin{aligned}
\operatorname{res}(h, v_0) &= \lim_{v\to v_o} \frac{(v - v_0)\mathrm{e}^{-|\Lambda| v}}{\mathrm{e}^{-i\alpha\pi} v^\alpha - \mathrm{e}^{i\psi}} \\
&= -\frac{1}{\alpha} \mathrm{e}^{i\psi(\frac{1}{\alpha}-1)} \mathrm{e}^{|\Lambda| \exp(i\frac{\psi}{\alpha})} \\
&= -\frac{1}{\alpha} \mathrm{e}^{i\psi(\frac{1}{\alpha}-1)} \mathrm{e}^{\Lambda}.
\end{aligned}$$

When $(R \to \infty)$, thus we have

$$\mathcal{I}_1 = \frac{1}{\alpha} \lambda^{\frac{1}{\alpha}-1} \mathrm{e}^{\Lambda} + \frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i} \int_0^\infty h(v)\,\mathrm{d}v + O\left(\frac{1}{n^{2-\alpha}|\Lambda|^{2\alpha}}\right).$$

Assume that $\lambda \in R_1$. Then v_0 is not enclosed by T because, in this case, we must have $-\delta \leqslant \arg v_0 < 0$, obtaining

$$\mathcal{I}_1 = \frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i} \int_0^\infty h(v)\,\mathrm{d}v + O\left(\frac{1}{n^{2-\alpha}|\Lambda|^{2\alpha}}\right).$$

Finally, the result for $\mathcal{I}_2$ can be readily obtained. $\square$

Lemma 6.5 (Refer [1])**.** *For $\lambda \in S_1 \cup S_2$ we have*

$$\mathcal{I}_3 = O\left(\frac{1}{n}\right) \quad (n \to \infty)$$

uniformly in λ.

Proof. Integrating by parts we obtain

$$\mathcal{I}_3 = \frac{-1}{2(n+1)\pi i} \left(\left[\frac{z^{-(n+1)}}{a(z) - \lambda} \right]_{\vartheta_3} - \int_{\vartheta_3} \frac{a'(z) z^{-(n+1)}}{(a(z) - \lambda)^2}\,\mathrm{d}z \right).$$

Since $1 \notin \vartheta_3$ and z is bounded away from 1, the function a is continuous and differentiable over ϑ_3. Thus,

$$c_0 := \sup\left\{\frac{1}{|a(z)-\lambda|} : z \in \vartheta_3\right\} \quad \text{and} \quad c_1 := \sup\left\{\frac{|a'(z)|}{|a(z)-\lambda|^2} : z \in \vartheta_3\right\}$$

are constants not depending on λ. Now

$$\left|\left[\frac{z^{-(n+1)}}{a(z)-\lambda}\right]_{\vartheta_3}\right| \leqslant c_0\left[\frac{1}{|1+\varepsilon \mathrm{e}^{-i\varphi}|^{n+1}} + \frac{1}{|1+\varepsilon \mathrm{e}^{i\varphi}|^{n+1}}\right] \leqslant c_0,$$

and

$$\int_{\vartheta_3}\left|\frac{a'(z)z^{-(n+1)}}{(a(z)-\lambda)^2}\right| \mathrm{d}z \leqslant c_1 \int_{\vartheta_3} \mathrm{d}z \leqslant c_1\varepsilon(\mathrm{e}^{i\varphi} - \mathrm{e}^{-i\varphi}).$$

Then

$$|\mathcal{I}_3| \leqslant \frac{c_0}{2(n+1)\pi} + \frac{c_1}{2(n+1)\pi}\varepsilon(\mathrm{e}^{i\varphi} - \mathrm{e}^{-i\varphi}) = O\Big(\frac{1}{n}\Big),$$

as $n \to \infty$ uniformly in λ. □

6.3 Individual eigenvalues

The following result gives a partial proof of the conjecture (3.4) of Bogoya, Grudsky and Malysheva [1], there are not eigenvalues in S_3.

Theorem 6.6 (Refer [1]). *Suppose that $\Lambda \in \hat{S}_3$. If $\frac{1}{2} < \alpha < 1$ or if $0 < \alpha \leqslant \frac{1}{2}$ with $\psi > \frac{\pi}{2}$, then we cannot have eigenvalues of $T_n(a)$ in $S_3 \setminus (R_1 \cup R_2)$.*

Proof. In this proof, all the order terms work with ($n \to \infty$) uniformly in Λ. Suppose that $\lambda \in S_3 \setminus (R_1 \cup R_2)$ (equivalently $\psi \in (\alpha\pi, \pi] \cup (-\pi, -\alpha\pi)$) is an eigenvalue of $T_n(a)$. Using Lemmas 6.4 and 6.5, and (6.9) we obtain

$$0 = |\lambda|^{\frac{1}{\alpha}-1}\int_0^\infty \mathrm{e}^{-|\Lambda|v} b(v,\psi)\,\mathrm{d}v + \Delta_3(\Lambda, n)$$

where

$$b(v,\psi) := \frac{\mathrm{e}^{-i\alpha\pi}}{v^\alpha - \mathrm{e}^{i(\psi-\alpha\pi)}} - \frac{\mathrm{e}^{i\alpha\pi}}{v^\alpha - \mathrm{e}^{i(\psi+\alpha\pi)}}$$

and $\Delta_3(\Lambda, n) = O\Big(\frac{1}{n}\Big)$, which is equivalent to saying that

$$0 = \frac{|\Lambda|^{1-\alpha}}{2\pi i(n+1)^{1-\alpha}}(G(\Lambda,\psi) + \Delta_4(\Lambda, n)),$$

where

$$G(\Lambda,\psi) := \int_0^\infty \mathrm{e}^{-|\Lambda|v} b(v,\psi)\,\mathrm{d}v$$

and $\Delta_4(\Lambda, n) = O\Big(\frac{1}{n^\alpha}\Big)$. Our aim is to show that $G(\cdot,\psi)$ has no zeros, since then (by the Rouche's theorem) $G(\cdot,\psi) - \Delta_4(\cdot, n)$ has no zeros either, getting a contradiction. Note that

$$\mathrm{e}^{i\varphi} b(v,\psi) = \frac{v^\alpha(\mathrm{e}^{i(\psi-\alpha\pi)} - \mathrm{e}^{i(\psi+\alpha\pi)})}{(v^\alpha - \mathrm{e}^{i(\psi-\alpha\pi)})(v^\alpha - \mathrm{e}^{i(\psi+\alpha\pi)})} \quad \text{and} \quad \frac{2\sin(\alpha\pi)}{i} = \mathrm{e}^{-i\psi}(\mathrm{e}^{i(\psi-\alpha\pi)} - \mathrm{e}^{i(\psi+\alpha\pi)}).$$

Then

$$\frac{i\mathrm{e}^{i\psi}G(\Lambda,\psi)}{2\sin(\alpha\pi)} = \int_0^\infty \frac{v^\alpha \mathrm{e}^{-|\Lambda|v}\mathrm{e}^{-i\psi}\kappa(v,\psi)}{|v^\alpha - \mathrm{e}^{i(\alpha\pi+\psi)}|^2|v^\alpha - \mathrm{e}^{-i(\alpha\pi-\psi)}|^2}\,\mathrm{d}v$$

where

$$\begin{aligned}\kappa(v,\psi) &:= (\mathrm{e}^{i\psi}v^\alpha - \mathrm{e}^{-i\alpha\pi})(\mathrm{e}^{i\psi}v^\alpha - \mathrm{e}^{i\alpha\pi}) \\ &= \mathrm{e}^{i\psi}(\mathrm{e}^{i\psi}v^{2\alpha} + \mathrm{e}^{-i\psi} - 2v^\alpha\cos(\alpha\pi)).\end{aligned}$$

We thus have

$$\mathfrak{Re}\bigg(\frac{i\mathrm{e}^{i\psi}G(\Lambda,\psi)}{2\sin(\alpha\pi)}\bigg) = \int_0^\infty \frac{v^\alpha \mathrm{e}^{-|\Lambda|v}\mathfrak{Re}(\mathrm{e}^{-i\psi}\kappa(v,\psi))}{|v^\alpha - \mathrm{e}^{i(\alpha\pi+\psi)}|^2|v^\alpha - \mathrm{e}^{-i(\alpha\pi-\psi)}|^2}\,\mathrm{d}v. \tag{6.30}$$

If for some Λ and ψ the equation $G(\Lambda,\psi) = 0$ is satisfied, then the integral in (6.30) will have a zero. Note that

$$\mathfrak{Re}(\mathrm{e}^{-i\psi}\kappa(v,\psi)) = \cos\psi\bigg(\Big(v^\alpha + \frac{\cos(\alpha\pi)}{\cos\psi}\Big)^2 + 1 - \frac{\cos^2(\alpha\pi)}{\cos^2\psi}\bigg).$$

If $\frac{1}{2} < \alpha < 1$, then (see Figure 3.2), $|\psi| \geqslant \alpha\pi > \frac{\pi}{2}$ and hence $\frac{\cos^2(\alpha\pi)}{\cos^2\psi} < 1$ and $\cos\psi < 0$, which shows that $\mathfrak{Re}(\mathrm{e}^{-i\psi}\kappa(v,\psi)) < 0$, making the integrand in (6.30) strictly negative, which yields the theorem in this case. If $0 < \alpha \leqslant \frac{1}{2}$ and $|\psi| > \frac{\pi}{2}$, then a similar analysis applies and we get the theorem in this case also. $\square$

Proof of Theorem 3.4. Let m, M be constants (depending only on the symbol a) satisfying $m \leqslant |\Lambda| \leqslant M$. In this proof all the order terms work with $n \to \infty$ uniformly in λ. Suppose that $\lambda \in (S_1 \setminus R_1) \cup (S_2 \setminus R_2)$. Using Lemmas 6.2 part (i), 6.4, and 6.5 in the equation (6.8) we get that λ is an eigenvalue of $T_n(a)$ if and only if

$$\frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\mathrm{e}^{\Lambda}\left(1+O\left(\frac{1}{n}\right)\right) = \frac{|\lambda|^{\frac{1}{\alpha}-1}}{2\pi i}\int_0^\infty \mathrm{e}^{-|\Lambda| v} b(v,\psi)\,\mathrm{d}v + O\left(\frac{1}{n}\right),$$

where

$$b(v,\psi) \coloneqq \frac{\mathrm{e}^{-i\alpha\pi}}{v^\alpha - \mathrm{e}^{i(\psi-\alpha\pi)}} - \frac{\mathrm{e}^{i\alpha\pi}}{v^\alpha - \mathrm{e}^{i(\psi+\alpha\pi)}}.$$

Noticing that

$$\frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\mathrm{e}^{\Lambda} O\left(\frac{1}{n}\right) = O\left(\frac{|\Lambda|^{1-\alpha}|\mathrm{e}^{\Lambda}|}{n^{2-\alpha}}\right) = O\left(\frac{1}{n^{2-\alpha}}\right),$$

$$O\left(\frac{1}{n|\lambda|^{\frac{1}{\alpha}-1}}\right) = O\left(\frac{n^{1-\alpha}}{n|\Lambda|^{1-\alpha}}\right) = O\left(\frac{1}{n^\alpha}\right),$$

then

$$\frac{2\pi i}{\alpha}\mathrm{e}^{i\psi(\frac{1}{\alpha}-1)}\mathrm{e}^{\Lambda} + O\left(\frac{1}{n^{2-\alpha}}\right) = \int_0^\infty \mathrm{e}^{-|\Lambda| v} b(v,\psi)\,\mathrm{d}v + O\left(\frac{1}{n^\alpha}\right),$$

we get the theorem in this case. Finally, suppose that $|\Lambda| \to 0$. Using the part (ii) of the Lemmas 6.2 and 6.3 in (6.8) we get that λ is an eigenvalue of $T_n(a)$ if and only if

$$\begin{aligned} 0 &= \lim_{|\Lambda|\to 0}\left(-\frac{\operatorname{res}(g, z_\lambda)}{\lambda^{\frac{1}{\alpha}-1}} + \frac{1}{\lambda^{\frac{1}{\alpha}-1}}(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3)\right) \\ &= -\lim_{|\Lambda|\to 0}\frac{\operatorname{res}(g, z_\lambda)}{\lambda^{\frac{1}{\alpha}-1}} + \lim_{|\Lambda|\to 0}\frac{2\mathrm{e}^{i\alpha\pi}\Gamma(\alpha-1)}{|\Lambda|^{1-\alpha}\mathrm{e}^{i\psi(\frac{1}{\alpha}-1)}}(1+o(1)) + \lim_{n\to\infty} O\left(\frac{1}{n}\right) \\ &= \infty, \end{aligned}$$

thus, we don't get eigenvalues in this case. □

Proof of Corollary 3.5. Considering the variable change $v = u\mathrm{e}^{i\frac{\psi}{\alpha}}$ in Theorem 3.4, as $n \to \infty$ uniformly in λ, we obtain

$$\frac{2\pi i}{\alpha}\mathrm{e}^{\Lambda} = \int_D \mathrm{e}^{-\Lambda u}\beta(u)\,\mathrm{d}u + O\left(\frac{1}{n^\alpha}\right),$$

where

$$\beta(u) := \frac{1}{u^\alpha \mathrm{e}^{i\alpha\pi} - 1} - \frac{1}{u^\alpha \mathrm{e}^{-i\alpha\pi} - 1}$$

and the integration path D is the straight line from 0 to $\infty \mathrm{e}^{-i\frac{\psi}{\alpha}}$.

Assume that $\lambda \in S_1$, then there exists a small constant μ satisfying $\frac{\alpha\pi}{2} - \alpha\mu < \psi < \alpha\pi$ and hence $-\pi < -\frac{\psi}{\alpha} < -\frac{\pi}{2} + \mu$. In order to make the integration path independent of λ, we make a path rotation by integrating over the triangle T with vertices 0, $\infty \mathrm{e}^{-i\frac{\psi}{\alpha}}$, and $\infty \mathrm{e}^{-i\frac{3}{4}\pi}$.

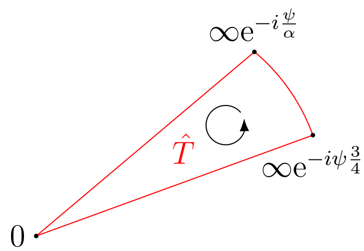


Figure 6.9: Contour $\hat{T}$

Since the singularities of β are $u = \mathrm{e}^{\pm i\pi}$, the integrand $\mathrm{e}^{-\Lambda u}\beta(u)$ is analytic on $\hat{T}$ and, moreover, the corresponding integral over the segment joining $\infty \mathrm{e}^{-i\frac{\psi}{\alpha}}$ and $\infty \mathrm{e}^{-i\frac{3}{4}\pi}$ is clearly 0 since $\mathrm{e}^{-\Lambda u}\beta(u) = o(1)$.

We thus have

$$\frac{2\pi i}{\alpha}\mathrm{e}^{\Lambda} = \int_{D_1} \mathrm{e}^{-\Lambda u}\beta(u)\,\mathrm{d}u + O\left(\frac{1}{n^\alpha}\right),$$

where D_1 is the straight line from 0 to $\infty \mathrm{e}^{-i\frac{3}{4}\pi}$. Finally, If $\lambda \in S_2$, a similar calculation produces

$$\frac{2\pi i}{\alpha}\mathrm{e}^{\Lambda} = \int_{D_2} \mathrm{e}^{-\Lambda u}\beta(u)\,\mathrm{d}u + O\left(\frac{1}{n^\alpha}\right),$$

where D_2 is the straight line from 0 to $\infty \mathrm{e}^{i\frac{3}{4}\pi}$. $\square$

Proof of Theorem 3.6. Suppose that Λ_s for $1 \leqslant s \leqslant k$ with $k \ll n$, are the zeros of F located in $\hat{S}_1 \cup \hat{S}_2$ and satisfying $F'(\Lambda_s) \neq 0$ for each s (i.e. each Λ_s is simple). We can pick a neighborhood U_s for each Λ_s with continuous and smooth boundary ∂U_s satisfying $|F(\cdot)| > |\Delta_2(\cdot, n)|$ over ∂U_s. In this case the Rouché Theorem 2.15 says that $F(\cdot) - \Delta_2(\cdot, n)$ must have a zero $\hat{\Lambda}_s$ in U_s. By Corollary 3.5, we know that each $\hat{\Lambda}_s$ corresponds to an eigenvalue $\lambda_j^{(n)}$ of $T_n(a)$. If necessary, re-enumerate Λ_s in order to get $s = j$. To prove the theorem, note that

$$F(\hat{\Lambda}_j) - F(\Lambda_j) = \Delta_2(\hat{\Lambda}_j, n) = F'(\Lambda_j)(\hat{\Lambda}_j - \Lambda_j) + O(|\hat{\Lambda}_j - \Lambda_j|^2),$$

which produces

$$O\left(\frac{1}{n^\alpha}\right) = (\hat{\Lambda}_j - \Lambda_j)\Big(F'(\Lambda_j) + O(|\hat{\Lambda}_j - \Lambda_j|)\Big).$$

By hypothesis we have $F'(\Lambda_j) \neq 0$, dividing both sides of the equation by second parentheses, we get

$$\hat{\Lambda}_j - \Lambda_j = (n+1)(\lambda_j^{(n)})^{\frac{1}{\alpha}} - \Lambda_j = O\left(\frac{1}{n^\alpha}\right).$$

Finally, solving for $\lambda_j^{(n)}$ then

$$\lambda_j^{(n)} = \left(\frac{\Lambda_j + O\left(\frac{1}{n^\alpha}\right)}{n+1}\right)^\alpha = \left(\frac{\Lambda_j}{n+1}\right)^\alpha \left(1 + O\left(\frac{1}{n^\alpha}\right)\right) \text{ as } n \to \infty \text{ uniformly in } \Lambda.$$

$\square$

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