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GAUSS-BONNET THEOREM AND ITS APPLICATIONS	
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Rodríguez Ramírez	Juan Camilo
DIRECTOR (ES) TESIS DOCTORAL O DEL TRABAJO DE GRADO	
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Son los términos que definen los temas que identifican el contenido. (*En caso de duda para designar estos descriptores, se recomienda consultar con la Sección de Desarrollo de Colecciones de la Biblioteca* Alfonso Borrero Cabal S.J *en el correo* biblioteca@javeriana.edu.co, *donde se les orientará).*

ESPAÑOL	INGLÉS
Geometría Riemanniana	**Riemannian geometry**
Teorema de Gauss-Bonnet	**Gauss-Bonnet Theorem**
Caracterísitica de Euler	**Euler characteristic**
Curvatura	**Curvature**
Formas diferenciales	**Differential forms**

RESUMEN DEL CONTENIDO EN ESPAÑOL E INGLÉS

(Máximo 250 palabras - 1530 caracteres)

Este trabajo tiene como fin estudiar el caso general del teorema de Gauss- Bonnet para superficies compactas orientadas sin frontera conocido como el teorema de Gauss-Bonnet-Chern el cual provee una relación fundamental entre la geometría del espacio y su topología. Para ello se estudiaron temas tales como variedades Riemannianas, formas diferenciales en variedades, característica de Euler, entre otros.

This thesis work is devoted to study the general case of the Gauss-Bonnet Theorem for compact oriented surfaces without boundary known as the Gauss-Bonnet-Chern Theorem which provides a fundamental relationship between the geometry of the space and its topology. For that some topics frequently used in this area were studied, we can mention Riemannian manifolds, differential forms on manifold and with values in a Lie algebra, Euler characteristic, among others.

Pontificia Universidad
JAVERIANA
Colombia

GAUSS–BONNET THEOREM AND APPLICATIONS

Juan Camilo Rodríguez Ramírez

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Directed by:
Andrés Vargas Dominguez Ph.D.

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Contents

To my family, my dog and Andrés.

Introduction

The german mathematician Carl Friedrich Gauß is the founder of the branch of mathematics known as differential geometry; he introduced the main concepts in his work *Disquisitiones generales circa superficies curvas* (1828) as a result of his observations concerning the characteristics of a curved surface that were part of his studies in geodesy. Originally, the Gauß–Bonnet theorem provides a fundamental relationship between the topology and the geometry of compact oriented surfaces in $\mathbb{R}^3$. In fact, it establishes that the integral of the Gaussian curvature K of the regular surface $M \subset \mathbb{R}^3$ equal to a constant multiplied by the Euler characteristic $\chi(M)$; this is, $\int_M K\, dA = 2\pi\chi(M)$. In 1920, Hopf thought about generalizing this formula by defining the degree of the normal (or Gauss) map $\widehat{N} : M \to S^{n-1}$ from an $(n-1)$-dimensional manifold in $\mathbb{R}^n$ to the unit sphere S^{n-1}. The degree of $\widehat{N}$, denoted by $\deg(\widehat{N})$, is $\frac{\int_M k\, dV}{\mathrm{vol}(S^{n-1})}$. Then for hypersurfaces, Hopf proved that $\chi(M) = \frac{1}{2}\deg(N)$. Later, Allendoerfer and Weil extended the Gauß–Bonnet formula to any even dimension in the case of a submanifold of the Euclidean space, this proof can be found in [16]. Finally, Chern generalized this theorem to any even dimension using Allendoerfer and Weil's formula without assuming that the manifold M was a submanifold of an Euclidean space using an intrinsic approach.

We will present the basic concepts of differential geometry which are necessary to study in detail the general version of the Gauß–Bonnet Theorem on manifolds, known as the Gauß–Bonnet–Chern Theorem. This is related to other important results in differential geometry on manifolds; for this, it is was necessary to learn a lot of tools and concepts frequently used in this area. Explicitly, we can mention differential forms on manifolds, connections, curvature, vector bundles, characteristic classes, among others. In addition, some calculations will be done to exemplify the mentioned Gauß–Bonnet–Chern theorem. Next, we will develop the related preliminary topics in the first chapter. After, chapter two is devoted to differential forms and its operations. In chapter three we study the proof of the mentioned Gauß–Bonnet-Chern Theorem and its preliminary steps. Finally, in chapter four, we give some examples of the former theorem.

Chapter 1

Preliminaries

In this section we shall give the basic concepts related to a Riemannian manifold and its topology. The former topic is based on [7], [12], and [14], and the latter is based on [17].

1.1 Topology and Euler characteristic

Here we will develop the topological concepts that are related to the Gauß–Bonnet theorem that we will prove.

Definition 1. Let $J \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. The *convex hull* of J, denoted by $\mathrm{conv}(J)$, is the smallest convex set containing J. i.e.

$$\mathrm{conv}(J) = \bigcap \{C \subset \mathbb{R}^n \mid J \subset C \text{ and } C \text{ is convex}\}.$$

Another important concept is the notion of affinely independence. It is defined as follows.

Definition 2. A set of points $\{p_i\}_{i=0}^m \subset \mathbb{R}^n$ with $m < n$ is said to be *affinely independent* if the equation

$$\sum_{i=0}^m a_i p_i = 0 \quad \text{and} \quad \sum_{i=0}^m a_i = 0$$

holds if and only if each $a_i = 0$.

Now we shall define what will allow us to triangulate manifolds.

Definition 3. Let X be a collection of m affinely independent points in $\mathbb{R}^n$. The *m-simplex* $\langle S \rangle$ spanned by the points $\{x_i\} \subset X$ is the set $\mathrm{conv}(X)$. The points in X are the vertices of the simplex, and a face of $\langle S \rangle$ is a simplex spanned by a nonempty subset of X. The dimension of $\langle S \rangle$ is m.

For example, a 0-simplex is a point, a 1-simplex is a line, and a 2-simplex is a triangle.

Definition 4. Let $K \subset \mathbb{R}^n$ be a collection of simplices in $\mathbb{R}^n$. then K is said to be a *simplicial complex* if it satisfies the following properties:

1. If a simplex is in K, then all of its faces are in K.
2. If $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are simplices such that $\langle S_1 \rangle \cap \langle S_2 \rangle \neq \emptyset$, then $\langle S_1 \rangle \cap \langle S_2 \rangle$ is a face of $\langle S_1 \rangle$ and $\langle S_2 \rangle$.

We remark that the dimension of K is the dimension of the highest-dimensional simplex in K, and also we should say that K carries the induced topology of $\mathbb{R}^n$. Now we can define what a triangulation is.

Definition 5. Let M be an n-dimensional manifold. A simplicial complex K is said to **triangulate** M if there is a homeomorphism $h : K \to M$.

Definition 6. Let K and $K^{'}$ be two simplicial complexes in $\mathbb{R}^n$. We say that the simplicial complex $K^{'}$ subdivides K if the union of all the simplexes in K equals the union of all the simplexes in $K^{'}$, and if every simplex in $K^{'}$ is a subset of a simplex in K.

There are some equivalent ways to define the Euler characteristic of a manifold M. One of those is to sum up the number of simplices of the same dimension together with a constant as follows:

Definition 7. Let K be a simplicial complex in $\mathbb{R}^n$. The Euler characteristic of K is the integer

$$\chi(M) = \sum_{i=0}^{n} -(1)^i \beta_i.$$

Where β_i denotes the number of simplices of dimension i.

Definition 8. Let M be an n-dimensional manifold. The **Euler characteristic** of M, denoted by $\chi(M)$ is the number $\chi(M) = \chi(K)$, where K is any simplicical complex that triangulates M.

To conclude this chapter, there are two key facts related to the Euler characteristic taken from [14]. The fist one is known as the Poincaré duality theorem. It is stated above.

Theorem 9. For a connected, compact oriented n-dimensional differentiable manifold, the bilinear map

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R}$$
$$([\omega], [\eta]) = \int_M \omega \wedge \eta$$

is non-degenerated i.e. there is an isomorphism between $H^{n-k}(M)$ and $H^k(M)^*$.

Theorem 10. The Euler characteristic of an odd-dimensional closed manifold is 0.

1.2 Vector bundles and connections

Let M be a C^∞-manifold. The set of all tangent spaces of M at any point is called the *tangent bundle*; this is

$$TM = \bigsqcup_{p \in M} T_p M$$

It comes with a natural projection

$$\pi : TM \to M$$

that maps each vector $X_p \in T_p M$ to p; and therefore $\pi^{-1}(p) = T_p M$. The tangent bundle admits a natural structure of a C^∞-manifold. Given a C^∞-atlas $\mathcal{S} = \{(U_\alpha, \varphi_\alpha)\}$ on a manifold M^n, for any tangent vector $v \in T_p U$, its image can be written in the form $\varphi_*(v) = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}$ where $\{\frac{\partial}{\partial x_j}\}_{j=1}^n$ is the basis of $T_0 \mathbb{R}^n$. Next, we define a bijective map $\tilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \in \mathbb{R}^{2n}$ by

$$\tilde{\varphi}(v) = (\varphi(p), a_1, \dots, a_n) \in \varphi(U) \times \mathbb{R}^n.$$

After this, a topology on TM is introduced saying that an open set on TM corresponds to the set $\pi^{-1}(U)$ and $\tilde{\varphi}$ is a homeomorphism. Then, $\tilde{S} = \{(\pi^{-1}(U), \tilde{\varphi}) \mid (U, \varphi) \in S\}$ is the corresponding atlas for the tangent bundle. Finally, the transition maps or coordinate transformations are all of class C^∞; thus, TM has a structure of a C^∞-manifold.

In the general case, a *real vector bundle* over a C^∞-manifold is a triple $\xi = (E, \pi, M)$ in which π is a C^∞ map from a C^∞-manifold E onto M such that:

- **Fiberwise linear structure** For each $p \in M$, the fiber $\pi^{-1}(p)$ has the structure of an n-dimensional real vector space.

- **Local triviality** For each $p \in M$ there is an open neighborhood U and a diffeomorphism $\varphi_U : \pi^{-1}(U) \cong U \times \mathbb{R}^n$ such that for each $q \in M$ the restriction $\pi^{-1}(q)$ provides a linear isomorphism $\varphi_U : \pi^{-1}(q) \cong \{q\} \times \mathbb{R}^n$.

The definition still works if $\mathbb{R}$ is replaced with the field $\mathbb{C}$; in this case, the bundle is called a *complex vector bundle*.

Definition 11. Given a vector bundle $\pi : E \to M$, a C^∞-map $s : M \to E$ such that $\pi \circ s = Id_M$ is called a *section*. In other words, a section assigns to each $p \in M$ a point $s(p)$ in the fiber $E_p = \pi^{-1}(p)$ over p. The set of sections of a vector bundle is denoted by $\Gamma(E)$.

We remark that sections of the tangent bundle TM are simply vector fields on the base space M. Therefore, we define the set of all vector fields on M as $\mathfrak{X}(M) := \Gamma(TM)$.

Definition 12. Let (E, π, M) be a smooth vector bundle over a Riemannian manifold (M, g). A *connection* on E is a map $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M)$ such that

- $\nabla_X(\lambda \cdot v + \mu \cdot w) = \lambda \cdot \nabla_X(v) + \mu \cdot \nabla_X(w)$,
- $\nabla_X(f \cdot v) = X(f) \cdot v + f \cdot \nabla_X(v)$,
- $\nabla_{f \cdot X + g \cdot Y}(v) = f \cdot \nabla_X(v) + g \cdot \nabla_Y(v)$,

for all $\lambda, \mu \in \mathbb{C}$, $X, Y \in \mathfrak{X}(M)$, $v, w \in \Gamma(E)$, and ∇_X denotes the contraction by a vector field X. Moreover, a connection over the tangent bundle E is said to be *compatible* with the Riemannian metric g (denoted simply by $\langle \cdot, \cdot \rangle$) if it satisfies

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

For all $X, Y, Z \in \mathfrak{X}(M)$. It is also required that the connection be **torsion-free** which means that

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

Taking advantage from the above relations, we now define the so-called Levi-Civita connection. First of all, from the property of compatibility with the metric g, we can write

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle Y, X\rangle = {} & \langle \nabla_X Y, Z\rangle + \langle Z, \nabla_Y X\rangle + \langle Y, \nabla_X Z\rangle \\ & - \langle Y, \nabla_Z X\rangle + \langle \nabla_Y Z, X\rangle - \langle \nabla_Z Y, X\rangle. \end{aligned}$$

Moreover, note that $\langle \nabla_X Y, Z\rangle + \langle \nabla_Y X, Z\rangle$ equals $2\langle \nabla_X Y, Z\rangle - \langle [X, Y], Z\rangle$; thus

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle Y, X\rangle = {} & 2\langle \nabla_X Y, Z\rangle + \langle \nabla_X Z - \nabla_Z X, Y\rangle \\ & + \langle \nabla_Y Z - \nabla_Z Y, X\rangle - \langle [X, Y], Z\rangle. \end{aligned}$$

The previous equation is the key to define the desired connection. Indeed, the *Levi-Civita* connection is the only connection on the tangent bundle for which the following equation holds:

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = {} & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle Z, [X, Y]\rangle \\ & - \langle Y, [Z, X]\rangle - \langle X, [Y, Z]\rangle. \end{aligned}$$

Finally, there is a useful tool known as the Christoffel symbols that we will need soon.

Definition 13. Let M be a Riemannian manifold with Levi-Civita connection ∇. Let (U, φ) be a local coordinate on M and $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ a local frame of TM on U. The *Christoffel symbols* $\Gamma_{ij}^k : U \to \mathbb{R}$ of the connection ∇ with respect to the local frame are given by

$$\sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial}{\partial x_j} \right).$$

Since the basis $\left\{\frac{\partial}{\partial x_i}\right\}$ is a coordinate one, then $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$, and in this case Γ_{ij}^k are also called as the Christoffel symbols of the first kind.

Curvature

It is also important to define the Riemannian curvature tensor, which can be seen as an endomorphism, as follows:

$$\mathrm{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(\mathrm{End}\, E)$$

Given by

$$\mathrm{R}(X,Y)\psi = [\nabla_X, \nabla_Y]\psi - \nabla_{[X,Y]}\psi,$$

where $X, Y \in \mathfrak{X}$, $\psi, \mathrm{R}(X,Y)\psi \in \Gamma(E)$, and $[\nabla_X, \nabla_Y]\psi := \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi$. The curvature R satisfies the following symmetries.

Proposition 14. Let (M, g) be a Riemannian manifold. For vector field $X, Y \in \mathfrak{X}(M)$, $\psi, \phi \in \Gamma(\mathrm{End}\, E)$ on M, we have

1. $\mathrm{R}(X,Y)\psi = -\mathrm{R}(Y,X)\psi$,
2. $\langle \mathrm{R}(X,Y)\psi, \phi\rangle = -\langle \mathrm{R}(X,Y)\phi, \psi\rangle$,
3. $\mathrm{R}(X,Y)\psi + \mathrm{R}(\psi, X)Y + \mathrm{R}(Y,\psi)X = 0$,
4. $\langle \mathrm{R}(X,Y)\psi, \phi\rangle = \langle \mathrm{R}(\psi,\phi)X, Y\rangle$.

Definition 15. Let (M, g) be a Riemannian manifold; let $p \in M$ be a point in M. Let $\mathrm{span}(X,Y) = \Pi_{XY}$ be the plane generated by the vectors X and Y, the quantity

$$\mathrm{K}_p(\Pi_{XY}) = \frac{\langle \mathrm{R}(X,Y)Y, X\rangle}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

is called the *sectional curvature* at p. When X, Y belong to an orthonormal basis of the tangent space at the point $p \in M$, the expression above reads simply,

$$\mathrm{K}_p(\Pi_{XY}) = \langle \mathrm{R}(X,Y)Y, X\rangle.$$

Note that when M has dimension 2, the sectional curvature is precisely the Gaussian curvature of the surface.

Product of manifolds

We shall take a look at the product manifold. Let (M_1, g_1) and (M_2, g_2) be two C^∞ Riemannian manifolds and consider the cartesian product $M_1 \times M_2$ between M_1 and M_2. Let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the natural projections. Then for $(p, q) \in M_1 \times M_2$ we introduce the Riemannian metric on the product manifold

$$g_{(p,q)} : T_{(p,q)}\,(M_1 \times M_2) \times T_{(p,q)}\,(M_1 \times M_2) \to \mathbb{R}, \quad \text{given by,}$$
$$g(X, Y) = g_1((\pi_1)_* X, (\pi_1)_* Y) + g_2((\pi_2)_* X, (\pi_2)_* Y).$$

Where $(\pi_1)_* X, (\pi_1)_* Y \in T_p M_1$ and $(\pi_2)_* X, (\pi_2)_* Y \in T_q M_2$. Then, we define a connection on the product manifold in the same way. Let ∇^1 and ∇^2 be the connection on M_1 and M_2, respectively. Then the Riemannian connection on $M_1 \times M_2$ is given by

$$\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^1_{Y_1} X_1 + \nabla^2_{Y_2} X_2.$$

With $X_i, Y_i \in \Gamma(TM_i)$. Finally, since the curvature is determined by a connection, then it is easy to derive an expression for the curvature tensor R of the product manifold $M_1 \times M_2$. This expression is presented below. Let E_1 be a vector bundle over M_1 with connection ∇^1. Let E_2 be a vector bundle over M_2 with connection ∇^2, R_1 be the curvature tensor of E_1 with respect to ∇^1, and R_2 be the curvature tensor of E_2 with respect to the connection ∇^2. Then the curvature tensor of $M_1 \times M_2$ is a map

$$\mathrm{R} : \mathfrak{X}(M_1 \times M_2) \times \mathfrak{X}(M_1 \times M_2) \to \Gamma(\mathrm{End}(E_1 \times E_2)),$$

given by,

$$\begin{aligned} \mathrm{R}(X_1 + X_2, Y_1 + Y_2)(Z_1 + Z_2) &= [\nabla_{X_1+X_2}, \nabla_{Y_1+Y_2}](Z_1 + Z_2) - \nabla_{[X_1+X_2, Y_1+Y_2]}(Z_1 + Z_2) \\ &= ([\nabla^1_{X_1}, \nabla^1_{Y_1}]Z_1 - \nabla^1_{[X_1,Y_1]} Z_1) + ([\nabla^2_{X_2}, \nabla^2_{Y_2}]Z_2 - \nabla^2_{[X_2,Y_2]} Z_2) \\ &= R_1(X_1, Y_1)Z_1 + R_2(X_2, Y_2)Z_2, \end{aligned} \tag{1.1}$$

where each $X_i, Y_i \in \mathfrak{X}(M_i)$ and $Z_i, \mathrm{R}(X_i, Y_i)Z_i \in \Gamma(E_i)$.

Chapter 2

Differential Forms

Differential forms play a fundamental role in the study of the geometry of a manifold. They can be understood in different ways, for instance, as an object that measures if a set of vectors are linearly independent or as an object that applied to n vectors, measures its volume multiplied by some scalar. this latter is what we shall use in the Gauß–Bonnet theorem. The concept involved here can be deepened in [13], [14], and [5].

2.1 Differential forms on manifolds

Let V a vector space over $\mathbb{R}$ or $\mathbb{C}$. An algebra generated by the elements of V with unity 1 such that $X \wedge Y = -Y \wedge X$ for any $X, Y \in V$ is called an **exterior algebra** of V and denoted by $\Lambda^* V$.

We shall provide general definitions of k-differential forms over a vector space V in order to define differential forms on a general manifold without using local coordinates.

First, note in the above definition that if $X = Y$, then $X \wedge X = 0$ for any $X \in V$. Next, we can decompose the exterior algebra whenever $\dim V = n$ by the direct sum

$$\Lambda^* V = \bigoplus_{k=0}^{\infty} \Lambda^k V$$

where $\Lambda^k V$ represents the subspace of all elements of degree k. Let $e_1, \ldots, e_n$ be a basis of V. The corresponding basis of each subspace is given by monomials of the form

$$e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad \text{where} \quad 1 \leqslant i_1 < \cdots < i_k \leqslant n.$$

Then the dimension of this subspaces is $\binom{n}{k}$ as long as $k \leqslant n$, otherwise $\dim \Lambda^k = 0$ since in this case, some elements in the basis would be repeated. Also note that $\Lambda^0 V$ is nothing but the constants in the field and $\Lambda^1 V$ is the same V. Moreover, let V^* be the dual space of V, then the same construction of the exterior algebra can be given.

Definition 16. A multilinear map $\omega : \underbrace{V \times \cdots \times V}_{k-times} \to \mathbb{R}$ is called an *alternating form* if for any $X_i \in V$

$$\omega(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(k)}) = (-1)^{\mathrm{sgn}(\sigma)} \omega(X_1, X_2, \ldots, X_k).$$

Where $\sigma \in S_k$ is any element of the permutation group of k letters. The set of all this elements of degree k is denoted by $\mathcal{A}^k(V)$ which has a structure of vector space. Once again, we can construct the whole algebra

$$\mathcal{A}^* V = \bigoplus_{k=0}^{\infty} \mathcal{A}^k V.$$

It is also possible to define $\mathcal{A}^0 V = \mathbb{R}$ or $\mathbb{C}$. On the other hand, there are isomorphisms between the vector space of all alternating forms on V and the exterior algebra of the dual space of V. One of the possibilities is that for each k

$$\psi_k : \Lambda^k V^* \to \mathcal{A}^k(V)$$

by setting

$$\psi_k(\omega)(X_1, \ldots, X_k) = \frac{1}{k!} \det(\alpha_i(X_j))$$

For an element of $\Lambda^k v^*$ of the form $\omega = \alpha_1 \wedge \cdots \wedge \alpha_k$

Now, let $T_p^* M = \{\alpha : T_p M \to \mathbb{R}; linear\}$ be the dual space of the tangent space $T_p M$ at the point p. Consider its exterior algebra $\Lambda^* T_p^* M$. After this, we are able to define forms on manifolds as follows.

Definition 17. Let M be a C^∞-manifold. An object ω that assigns an element of class C^∞ $\omega_p \in \Lambda^* T_p^* M$ to each point $p \in M$ is called a **k-form** on M.

Now, we will deduce what form ω_p should have. Let $U \in M$ be a neighborhood, and let $\{x_i\}_{i=1}^n$ be coordinate functions on U, so for any point $p \in U$, the set

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$$

is a basis of the tangent space $T_p M$. We want to find the dual basis in order to write ω in each local coordinate, then think of each x_i as a C^∞ function $x_i : U \to \mathbb{R}$, and consider its differential $(dx_i)_p : T_p M \to T_{x_i(p)} \mathbb{R} \cong \mathbb{R}$ at the point p. Because of this, $(dx_i)_p$ can be seen as an element of the dual space $T_p^*(M)$; then $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{i,j}$; then $\{(dx_1)_p, (dx_2)_p, \ldots, (dx_n)_p\}$ is the desired basis of $T_p^* M$; for this, we can write ω_p as follows:

$$\omega_p = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}(p)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Where ω_p is of class C^∞ if each $f_{i_1 \cdots i_k}(p)$ is of class C^∞.

There exists an alternative point of view via vector bundles. Let T^*M be the cotangent bundle over M, this is, the dual space of the tangent bundle of M. As we have set up above; consider the exterior algebra of the cotangent space.

$$\Lambda^k T^*M = \bigsqcup_p \Lambda^k T_p^* M.$$

Here, it is possible to understand k-forms on M as sections of class C^∞ of $\Lambda^k T^*M$, then

$$\mathcal{A}^k(M) = \{\text{all } \ C^\infty\text{-sections of } \Lambda^k T^*M\}.$$

Equivalently, a k-form can be interpreted in the following way. Let ω be a k-form on M, that is, ω assigns an element ω_p of $\Lambda^* T^*M$ to each p; When all p are put together, ω can be understood as a multilinear alternating map:

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M).$$

Here, multilinear means that is also linear with respect to the multiplication of vector fields by functions (this because $\mathfrak{X}(M)$ is also a module over $C^\infty(M)$, then $fX \in \mathfrak{X}(M)$ for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$); and indeed,

$$\omega(X_1, \cdots, fX_i + gX{i}', \cdots, X_k) = f \cdot \omega(X_1, \cdots, X_i, \cdots, X_k) + g \cdot \omega(X_1, \cdots, X{i}', \cdots, X_k),$$

$$\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = (-1)^{\mathrm{Sgn}(\sigma)} \omega(X_1, \ldots, X_k).$$

2.2 Operations with differential forms

It is important now to know how one object of $\mathcal{A}^k(M)$ acts on one object of $\mathcal{A}^l(M)$. The product between two differential forms is known as **exterior product**, then

Definition 18. If ω has a local representation as $\omega_p = \sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is a k-differential form at the point p on M and η has its local representation given by $\eta_p = \sum_J g_J dx_{j_1} \wedge \cdots \wedge dx_{j_l}$ is an l-differential form at p on M; the exterior product between them is given by

$$(\omega \wedge \eta)_p = (\omega)_p \wedge (\eta)_p = \sum_{I,J} f_I \cdot g_J \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge d_{j_l},$$

where I and J denote two sets of k and l ordered indexes taken from n and each $dx_i = (dx_i)_p$.

The exterior product satisfies the following properties such as:

- $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega,$

- for any vector fields $X_1, \ldots, X_{k+l} \in \mathfrak{X}(M)$ one has

$$\omega \wedge \eta(X_1, \ldots, X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (-1)^{\mathrm{sgn}(\sigma)} \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})$$

  where σ is a permutation of $k+l$ letters taken from $\{1, \ldots, n\}$.

Another important operation applied to differential forms is known as *exterior differentiation*. This operation is a linear map

$$d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M),$$

That takes a k-differential form ω and maps it into

$$d\omega = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where ω is locally viewed as $\omega = f(x_1, x_2, \ldots, x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Moreover, d also satisfies the following properties

- $d \circ d = 0$,
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

Theorem 19. Let M be a C^∞-manifold and $\omega \in \mathcal{A}^k(M)$ a k-form on M. Then for any vector fields $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$, the following equation holds

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = \\ & \frac{1}{k+l} \Bigg(\sum_{i=1}^{k+1} (-1)^{i+1} X_i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \\ & + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \Bigg). \end{aligned} \tag{2.1}$$

Where $[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i$ and $\hat{X}_i$ means that X_i is omitted.

The last operation with differential forms we will introduce is a natural action of C^∞-maps on forms. Let $f : M \to N$ be a C^∞-map from a C^∞-manifold to N. Consider the differential $df = f_* : T_pM \to T_{f(p)}N$ at each point $p \in M$. This differential induces a dual map

$$\begin{aligned} & f^* : T^*_{f(p)}N \to T^*_pM, \quad \text{given by} \\ & f^*(\alpha)(X) = \alpha(f_*(X)), \end{aligned}$$

for $\alpha \in T^*_{f(p)}N$ and $X \in T_pM$. In turn, this induces a linear map between k-forms on M. let $f^* : \Lambda^k T^*_{f(p)}N \to \Lambda^k T^*_pM$ be such map for any k, then we have a homomorphism between the exterior algebras

$$f^* : \mathcal{A}^*(N) \to \mathcal{A}^*(M).$$

For a differential form $\omega \in \mathcal{A}^*(N)$ we call $f^*\omega \in \mathcal{A}^*(M)$ the **pullback** by f. Given $\{X_1, \ldots, X_k\} \in T_pM$ we compute

$$f^*\omega(X_1, \ldots, X_k) = \omega(f_*(X_1), \ldots, f_*(X_k)).$$

As the other operations presented, the pullback has important properties.

Let M and N be C^∞-manifolds. Let f be a C^∞ map between them and $f^* : \mathcal{A}^*(N) \to \mathcal{A}^*(M)$ the induced map by f, then f^* satisfies the following properties:

- $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$,
- $d(f^*\omega) = f^*(d\omega)$.

2.3 Vector-valued differential forms

Definition 20. Let (E, π, M) be a vector bundle over M. A section of $\Lambda^k T^*M \otimes E$ is called a **k-form on M with values in E**. So

$$\mathcal{A}^*(M; E) = \Gamma(\Lambda^k T^*M \otimes E).$$

As we have identified a k-form on M with a multilinear map from $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)$ to $C^\infty(M)$, we can identify $\mathcal{A}^k(M; E) = \{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \Gamma(E)\}$. Then any element of $\mathcal{A}^k(M; E)$ can be viewed as a linear combination of elements of the form $\alpha \otimes s$ where $\alpha \in \mathcal{A}^k(M)$ and $s \in \Gamma(E)$ which is $\mathcal{A}^0(M; E)$.
The exterior product $\mathcal{A}^k(M; E) \times \mathcal{A}^l(M; E) \to \mathcal{A}^{k+l}(M; E)$ between a k-form and an l-form into one $(k+l)$-form is also well-defined .

Keeping this in mind, we shall write the connection of a bundle and the curvature in terms of differential forms. Let ∇ be a connection in a vector bundle (E, π, M), and R its curvature. Let $U \in M$ be an open set in M, and let $s_1, \ldots, s_n \in \Gamma(E_U)$ be a frame field. For an arbitrary vector field $X\mathfrak{X}(U)$, we write:

$$\nabla_X s_i = \sum_{i=1}^{n} \omega^i_j(X) s_i.$$

With $\omega^i_j(X) \in C^\infty(U)$. Each ω^i_j can be considered as one 1-form on U because $\omega^i_j(fX) = f\omega^i_j(X)$. Moreover, all these 1-forms together define a 1-form over U with values in the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$; let $\omega = (\omega^i_j)$ be such form.

In the same way for arbitrary vector fields X, Y on $U \subset M$, there are 2-forms related to the curvature R as follows:

$$\mathrm{R}(X,Y)(s_j) = \sum_{i=1}^{n} \Omega^i_j(X,Y)s_i.$$

With $\Omega^i_j(X,Y) \in C^\infty(U)$. Then $\Omega = (\Omega^i_j)$ is a 2-form on U with values in the Lie algebra $\mathfrak{gl}(n;\mathbb{R})$. Finally, we say that ω is the **connection** 1**-form** and Ω is the **curvature** 2**-form**.

Chapter 3

Gauß–Bonnet Theorem

Let us start by presenting a preliminary step of the well-known Gauß–Bonnet theorem for oriented regular surfaces in $\mathbb{R}^3$ with boundary.

Theorem 21. Let M be an oriented regular surface in $\mathbb{R}^3$ i.e. an oriented regular 2-dimensional manifold with the induced metric of $\mathbb{R}^3$. Let $X : U \to M$ be a local parametrization of M such that $X(U)$ is simply connected. Let $\gamma : \mathbb{R} \to M$ parametrize a piecewise regular simple, closed, positively oriented curve on M by arclength. If $L \in \mathbb{R}^+$ is the period of γ then

$$\int_0^L \kappa_g(s)ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \int_{\mathrm{int}(\gamma)} KdA \tag{3.1}$$

Where K is the Gaussian curvature of M; $\kappa_g(s)$ is the geodesic curvature of γ which is defined as $\kappa_g(s) = \langle N(\gamma(s)) \times \dot{\gamma}(s), \ddot{\gamma}(s)\rangle$; N is the Gauss map (or normal map), and $\{\alpha_i\}_{i=1}^n$ are the inner angles at the corner points of the piecewise regular curve.

Proof. Let $\{e_1, e_2\}$ be the orthonormal basis which is obtained by applying the Gram–Schmidt process on $\{X_u, X_v\}$. Let us define an angle $\theta(s) : \mathbb{R} \to \mathbb{R}$ such that the unit tangent vector $\dot{\gamma}$ satisfies

$$\dot{\gamma}(s) = e_1(s)\cos\theta(s) + e_2(s)\sin\theta(s)$$

Then we have for $e_i = e_i(s)$ and $\theta = \theta(s)$

$$\begin{aligned} N \times \dot{\gamma} &= N \times (e_1 \cos\theta + e_2 \sin\theta) \\ &= -e_1 \sin\theta + e_2 \cos\theta, \end{aligned}$$

And for $\ddot{\gamma}$,

$$\ddot{\gamma} = (-e_1 \sin\theta + e_2 \cos\theta)\dot{\theta} + \dot{e_1}\cos\theta + \dot{e_2}\sin\theta.$$

Therefore, the geodesic curvature satisfies

$$\begin{aligned}\kappa_g &= \langle N \times \dot{\gamma}, \ddot{\gamma}\rangle \\ &= \langle -e_1 \sin\theta + e_2 \cos\theta, \dot{\theta}\{-e_1 \sin\theta + e_2 \cos\theta\}\rangle \\ &+ \langle -e_1 \sin\theta + e_2 \cos\theta, \dot{e_1} \cos\theta = \dot{e_2} \sin\theta\rangle \\ &= \dot{\theta} - \langle e_1, \dot{e_2}\rangle.\end{aligned}$$

Then when we integrate the geodesic curvature κ_g we obtain

$$\int_0^L \kappa_g ds = \int_0^L \dot{\theta} ds - \int_0^L \langle e_1, \dot{e_2}\rangle ds.$$

We will denote by $\partial_{\bullet} e_i$ the derivative of e_i in the direction of $\bullet$,then the last term equals

$$\begin{aligned}\int_0^L \langle e_1, \dot{e_2}\rangle ds &= \int_0^L \langle e_1, \dot{u}\partial_u e_2 + \dot{v}\partial_v e_2\rangle ds \\ &= \int_\alpha \langle e_1, \partial_u e_2\rangle du + \langle e_1, \partial_v e_2\rangle dv \\ &= \int_{Int(\alpha)} (\partial_u\langle e_1, \partial_v e_2\rangle - \partial_v\langle e_1, \partial_u e_2\rangle) du\, dv \qquad (*) \\ &= \int_{Int(\alpha)} (\langle \partial_u e_1, \partial_v e_2\rangle + \langle e_1, \partial_u\partial_v e_2\rangle - \langle \partial_v e_1, \partial_u e_2\rangle - \langle e_1, \partial_u\partial_v e_2\rangle) du\, dv \\ &= \int_{Int(\alpha)} (\langle \partial_u e_1, \partial_v e_2\rangle - \langle \partial_v e_1, \partial_u e_2\rangle) du\, dv \qquad (**) \\ &= \int_{int(\alpha)} K\sqrt{EG - F^2} du\, dv \\ &= \int_{int(\alpha)} K dA.\end{aligned}$$

In $(*)$ we have used Green's theorem, the proof of $(**)$ can be found in [6] (Pg. 51), and $I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is the first fundamental form (or metric) of the surface. Now we shall see what $\int_0^l \dot{\theta}(s) ds$ is:

$$\int_0^l \dot{\theta}(d) ds = \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \dot{\theta}(s) ds.$$

Where each $\{s_i\}$ is one of the corners of the piecewise curve γ. Thus, this integral measures the change of angle with respect to the orthonormal basis $\{e_1, e_2\}$ along each arc. The jumps at each corner are given by the angle $(\pi - \alpha_i)$, and at the end, after moving around the curve the total angle sums up 2π, thus

$$2\pi = \int_0^L \theta(\dot{s}) ds + \sum_{i=1}^n (\pi - \alpha_i)$$

$$\pi(n-2) + \sum_{i=1}^{n} \alpha_i = \int_0^L \dot{\theta}(s)ds.$$

□

3.1 Gauß–Bonnet theorem via moving frames

First, since this proof of the Gauß–Bonnet theorem involves changes of frames, we need to introduce the structure equations. The main ideas of this proof are summarized as follows:

1. Provide the structure equations in order to link two different frames through the connection form.
2. Write down the explicit change of frame. Intuitively, this change of frame is given by the differential of an angle form.
3. By integrating this form, we define the index of a vector field at a singularity.
4. After writing the connections as a multiple of the volume form; which is indeed, the Gaussian curvature, we use the Stokes Theorem to catch those singularities as the sum of the indexes in the 2-dimensional manifold.

3.2 Structure equations

Theorem 22. Let $\{e_i\}$ be a moving frame in an open set $U \in \mathbb{R}^n$. Let $\{\omega_i\}$ be its coframe and ω_j^i be the connection forms on U in the frame $\{e_i\}$. Then

$$d\omega_i = \sum_k \omega_k \wedge \omega_k^i, \tag{3.2}$$

$$d\omega_j^i = \sum_k \omega_k^i \wedge \omega_j^k, \quad i, j, k = 1, \ldots, n. \tag{3.3}$$

Proof. Let $\{a_i\}$ be the canonical basis of $\mathbb{R}^n$; let $x_i : U \in \mathbb{R}^n \to \mathbb{R}$ be the "projection" function that assigns to each point $p = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the i-th coordinate. Then, dx_i is a 1-form on U and of course, $dx_i(a_j) = \delta_{ij}$; thus $\{dx_i\}$ is the associated coframe of $\{a_i\}$. Then we write

$$e_i = \sum_j \beta_j^i a_j,$$

where each β_j^i is a C^∞-function. then from this:

$$de_i = \sum_j d\beta_j^i a_j. \tag{3.4}$$

We also can write de_i in a similar way

$$de_i = \sum_k \omega_k^i e_k.$$

By substituting e_i we get

$$de_i = \sum_k \omega_k^i e_k = \sum_k \omega_k^i \left(\sum_j \beta_j^i a_j \right) = \sum_{j,k} \omega_k^j \beta_j^i a_j. \tag{3.5}$$

Then using 3.4 and 3.5 we get

$$d\beta_j^i = \sum_k \omega_k^i \beta_j^k. \tag{3.6}$$

On the other hand, since there exists ω_i such that $\omega_i(e_j) = \delta_{ij}$, we also can write

$$\omega_i = \sum_j \beta_j^i dx_j, \qquad d\omega_i = \sum_j d\beta_j^i \wedge dx_j \tag{3.7}$$

Then, we get the first equation using 3.6 and 3.7

$$d\omega_i = \sum_j \left(\sum_k \omega_k^i \beta_j^k \right) \wedge dx_j = \sum_k \omega_k^i \wedge \omega_k$$

In the second equation it is necessary to differentiate $d\beta_j^i$; therefore

$$0 = d(d\beta_j^k) = \sum_k d\omega_k^i \beta_j^k - \sum_k \omega_k^i \wedge d\beta_k^j$$

Then

$$\sum_k d\omega_k^i \beta_j^k = \sum_k \omega_k^i \wedge d\beta_k^j$$

Substituting $d\beta_j^k$ we finally arrive at

$$\sum_k d\omega_k^i \beta_j^k = \sum_k \omega_k^i \wedge \left(\sum_s \omega_s^i \beta_j^s \right)$$

Multiplying by the inverse of (β_j^k) we got

$$d\omega_r^i = \sum_k \omega_k^i \wedge \omega_r^k.$$

□

It is time to provide the relationship between the connection forms ω_j^i with the metric of the manifold.

Proposition 23. For every vector field $X \in U$, we have

$$\omega_j^i(X) = \langle \nabla_X e_i, e_j \rangle. \tag{3.8}$$

Proof. First of all, we have to recall some concepts.

1. The Christoffel symbols of the second kind Γ_{ij}^k with respect to the orthonormal frame $\{e_1, \ldots, e_n\}$ are defined by

$$\nabla_{e_i} e_j = \Gamma_{ij}^k e_k.$$

   Thus the coefficient of e_k is $\langle \nabla_{e_i} e_j, e_k \rangle = \Gamma_{ij}^k$.

2. $\Gamma_{ij}^k = -\Gamma_{ik}^j$ because

$$\begin{aligned} 0 &= e_i \langle e_j, e_k \rangle \\ &= \langle \nabla_{e_i} e_j, e_k \rangle + \langle e_j, \nabla_{e_i} e_k \rangle \\ &= \Gamma_{ij}^k + \Gamma_{ik}^j. \end{aligned}$$

   We remark that $\Gamma_{ij}^j = 0$ for all j.

We get an important relation between ω_i and the connection forms ω_k^i through the structure equations. We shall show it for any element X of an orthonormal basis $\{e_i\}_{i=1}^n$ of $U \in M$. Then for any $1 \leq i, j, k \leq n$,

$$\begin{aligned} d\omega_i &= \omega_1^i \wedge \omega_1 + \cdots + \omega_n^i \wedge \omega_n \\ d\omega_i(e_j, e_k) &= (\omega_1^i \wedge \omega_1)(e_j, e_k) + \cdots + (\omega_n^i \wedge \omega_n)(e_j, e_k). \end{aligned}$$

All the terms equals to 0 except for:

$$\begin{aligned} d\omega_i(e_j, e_k) &= (\omega_j^i \wedge \omega_j)(e_j, e_k) + (\omega_k^i \wedge \omega_k)(e_j, e_k) \\ &= \omega_j^i(e_j)\omega_j(e_k) - \omega_j^i(e_k)\omega_j(e_j) + \omega_k^i(e_j)\omega_k(e_k) - \omega_k^i(e_k)\omega_k(e_l) \\ &= \omega_k^i(e_j)\omega_k(e_k) - \omega_j^i(e_k)\omega_j(e_j) \\ &= \omega_k^i(e_j) - \omega_j^i(e_k). \end{aligned}$$

Then we have

$$\omega_k^i(e_j) = d\omega_i(e_j, e_k) + \omega_j^i(e_k), \tag{3.9}$$

which implies

$$\omega_k^i(e_j) = e_j \omega_i(e_k) - e_k \omega_i(e_j) - \omega_i([e_j, e_k]) + \omega_j^i(e_k).$$

Now he have some cases. First if $i = j$, then

$$\begin{aligned}
\omega^i_k(e_i) &= e_i\omega_i(e_k) - e_k\omega_i(e_i) - \omega_i(\nabla_{e_i}e_k - \nabla_{e_k}e_i) + \omega^i_i(e_k) \\
&= -\omega_i(\Gamma^l_{ik}e_l - \Gamma^l_{ki}e_l) \\
&= -\Gamma^i_{ik} \\
&= \Gamma^k_{ii} \\
&= \langle \nabla_{e_i}e_i, e_k \rangle.
\end{aligned}$$

Second, if $i = k$, then

$$\begin{aligned}
0 &= e_j\omega_i(e_j) - e_i\omega_i(e_j) - \omega_i([e_j, e_i]) + \omega^i_j(e_i) \\
\omega_i(\nabla_{e_j}e_i - \nabla_{e_i}e_j) &= \omega^i_j(e_i) \\
\omega_i(\Gamma^l_{ji}e_l - \Gamma^l_{ij}e_l) &= \omega^i_j(e_i) \\
-\Gamma^i_{ij} &= \omega^i_j(e_i) \\
\Gamma^j_{ii} &= \omega^i_j(e_i) \\
\langle \nabla_{e_i}e_i, e_j \rangle &= \omega^i_j(e_i).
\end{aligned}$$

□

3.3 Change of frames

The change of frame is given by a differential form $\tau = \tilde{\omega_{12}} - \omega_{12}$ which is the differential of the angle between two different frames.

Let $\{e_1, e_2\}$ and $\{\tilde{e_1}, \tilde{e_2}\}$ be orthonormal frames on $U \in M$. Let $A(p)$ be the change of basis matrix from $\{e_1, e_2\}$ to $\{\tilde{e_1}, \tilde{e_2}\}$, which is $C^\infty(U)$. Note also that the determinant of A tell us if the orientation of the new basis changes or not. As $A \in O(n)$ is an orthogonal matrix, it should be of the form

$$\begin{aligned}
A(\theta)_p &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} f & g \\ -g & f \end{pmatrix}.
\end{aligned}$$

Then $\theta = \arctan\left(\frac{g}{f}\right)$, $f \neq 0$, is the angle "function". By differentiating it, we get

$$d\theta = d\left(\arctan\left(\frac{g}{f}\right)\right) = \frac{f\,dg - g\,df}{f^2 + g^2} := \tau.$$

If the rotation preserves the orientation, i.e. $\det(A) = 1$, then

$$\tau := f\,dg - g\,df.$$

Lemma 24. We have $\tilde{\omega}^i_j - \omega^i_j = \tau$, where τ is expressed in the above form.

Proof. First of all, we shall write each frame in term of the other; for this

$$\begin{aligned} A\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} &= \begin{pmatrix} f & g \\ -g & f \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ &= \begin{pmatrix} f\,e_1 + g\,e_2 \\ f\,e_2 - g\,e_1 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{e_1} \\ \tilde{e_2} \end{pmatrix}. \end{aligned}$$

In the same way we can take the inverse matrix of A and write each e_i in terms of each $\tilde{e_j}$ so that we get

$$\tilde{e_1} = f\,e_1 + g\,e_2 \tag{3.10}$$

$$\tilde{e_2} = f\,e_2 - g\,e_1 \tag{3.11}$$

$$e_1 = f\,\tilde{e_1} - g\,\tilde{e_2} \tag{3.12}$$

$$e_2 = f\,\tilde{e_2} + g\,\tilde{e_1}. \tag{3.13}$$

Using all of these equation we get the corresponding equations in the dual space.

$$\tilde{\omega_1} = f\,\omega_1 + g\,\omega_2 \tag{3.14}$$

$$\tilde{\omega_2} = f\,\omega_2 - g\,\omega_1 \tag{3.15}$$

$$\omega_1 = f\,\tilde{\omega_1} - g\,\tilde{\omega_2} \tag{3.16}$$

$$\omega_2 = f\,\tilde{\omega_2} + g\,\tilde{\omega_1}. \tag{3.17}$$

Now, using the [theorem 6], (3.12), (3.13), and the structure equations with ω_1 we have

$$\begin{aligned} d\omega_1 &= df \wedge \tilde{\omega}_1 + f\,d\tilde{\omega}_1 - dg \wedge \tilde{\omega}_2 - g\,d\tilde{\omega}_2 \\ &= df \wedge (f\,\omega_1 + g\,\omega_2) + f\,\tilde{\omega}^1_2 \wedge (f\,\omega_2 - g\,\omega_1) \\ &- dg \wedge (f\,\omega_2 - g\,\omega_1) - g\,\tilde{\omega}^2_1 \wedge (f\,\omega_1 + g\,\omega_2) \\ &= (f\,df + g\,dg) \wedge \omega_1 + (g\,df - f\,dg) \wedge \omega_2 + (f^2 + g^2)\tilde{\omega}^1_2 \wedge \omega_2. \end{aligned}$$

The first term is zero since $0 = \frac{1}{2}d(f^2 + g^2) = f\,df = g\,dg$. Then

$$\begin{aligned} d\omega_1 &= (g\,df - f\,dg) \wedge \omega_2 + (f^2 + g^2)\tilde{\omega}^1_2 \wedge \omega_2 \\ d\omega_1 &= -\tau \wedge \omega_2 + \tilde{\omega}^1_2 \wedge \omega_2 \\ d\omega_1 &= (\tilde{\omega}^1_2 - \tau) \wedge \omega_2. \end{aligned}$$

By the structure equations $d\omega_1 = \omega_2^1 \wedge \omega_2$; therefore

$$\tilde{\omega}_2^1 - \tau = \omega_2^1.$$

□

3.4 Curvature and index of a vector field

In this section we shall present the main concepts involved in the proof of the Gauß–Bonnet theorem.

Recall that the Gaussian curvature in terms of the Riemannian curvature tensor is given by

$$\mathrm{K}(p)(e_1, e_2) = \langle \mathrm{R}(e_1, e_2)e_2, e_1 \rangle,$$

where $\{e_1(p), e_2(p)\}$ is an arbitrary orthonormal basis for T_pM. The following theorem provides a relationship between the moving frame and the Gaussian curvature.

Theorem 25. If $\{e_1, e_2\}$ is a moving frame on $U \subset M$ and ω_2^1 is the connection form corresponding to the associated coframe $\{\omega_1, \omega_2\}$, then

$$d\omega_2^1 = -\,\mathrm{K}\,\omega_1 \wedge \omega_2. \tag{3.18}$$

Proof. It is enough to show that $d\omega_2^1(e_1, e_2) = -\,\mathrm{K}$ because $\omega_1 \wedge \omega_2(e_1, e_2) = 1$. Then we have

$$\begin{aligned} d\omega_2^1(e_1, e_2) &= e_1(\omega_2^1(e_2)) - e_2(\omega_2^1(e_1)) - \omega_2^1([e_1, e_2]) \\ &= e_1\langle \nabla_{e_2} e_1, e_2 \rangle - e_2\langle \nabla_{e_1} e_1, e_2 \rangle - \langle \nabla_{[e_1,e_2]} e_1, e_2 \rangle \\ &= \langle \nabla_{e_1} \nabla_{e_2} e_1, e_2 \rangle + \langle \nabla_{e_2} e_1, \nabla_{e_1} e_2 \rangle - \langle \nabla_{e_2} \nabla_{e_1} e_1, e_2 \rangle \\ &\quad - \langle \nabla_{e_1} e_1, \nabla_{e_2} e_2 \rangle - \langle \nabla_{[e_1,e_2]} e_1, e_2 \rangle. \end{aligned}$$

Note that $\langle \nabla_{e_1} \nabla_{e_2} e_1 - \nabla_{e_2} \nabla_{e_1} e_1 - \nabla_{[e_1,e_2]} e_1, e_2 \rangle = \langle \mathrm{R}(e_1, e_2)e_1, e_2 \rangle$; thus

$$\begin{aligned} d\omega_2^1 &= -\langle \mathrm{R}(e_1, e_2)e_2, e_1 \rangle + \langle \nabla_{e_2} e_1, \nabla_{e_1} e_2 \rangle - \langle \nabla_{e_1} e_1, \nabla_{e_2} e_2 \rangle \\ &= -\,\mathrm{K} + \langle \Gamma_{21}^k e_k, \Gamma_{12}^k e_k \rangle - \langle \Gamma_{11}^k e_k, \Gamma_{22}^k e_k \rangle \\ &= -\,\mathrm{K} + \langle \Gamma_{21}^2 e_2, \Gamma_{12}^1 e_1 \rangle - \langle \Gamma_{11}^2 e_2, \Gamma_{22}^1 e_1 \rangle \\ &= -\,\mathrm{K}\,. \end{aligned}$$

□

Definition 26. Let X be a vector field on M. Let $p \in M$ be a singularity of X, and C a simple closed curve on $U \subset M$. Then the number

$$\mathrm{Ind}_X(p) = \frac{1}{2\pi} \int_C \tau \tag{3.19}$$

is called the **index** of X at the point p.

This index represents the number of full turns that the vector field makes around p.

Lemma 27. If B_r is the disk of radius r, with respect to the Riemannian distance, centered at p and $S_r = \partial B_r$, then the limit

$$\lim_{r\to 0} \frac{1}{2\pi} \int_{S_r} \omega_2^1 = I$$

exists and equals $\mathrm{Ind}_X(p)$. Therefore, the definition of the index is independent of the choice of the reference frame $\{e_1, e_2\}$.

Theorem 28. For every compact orientable 2-dimensional manifold M and every smooth vector field X on M with only isolated singularities $p_1, \ldots, p_n$, we have

$$\int_M \mathrm{K}\, dA = 2\pi \sum_{i=1}^{n} \mathrm{ind}_X(p_i). \tag{3.20}$$

Proof. Let $U = M \setminus \{p_1, \ldots, p_n\}$. Then X is non-vanishing on U, so we can define $e_1 = \frac{X}{|X|}$. let e_2 be the unit vector field orthogonal to e_1 such that $\{e_1, e_2\}$ is an oriented orthonormal frame on U. Let $\{\omega_1, \omega_2\}$ be the associated coframe, and the corresponding connection forms by ω_j^i. Let B_{p_i} be the disk of radius $r_i > 0$ centered at p_i such that $B_{p_i} \cap B_{p_j} = \emptyset$ for all $i \neq j$. Now, define

$$V = M \setminus \bigcup_{i=1}^{n} B_{p_i}$$

Note here that each ∂B_{p_i} viewed as a boundary from V is the opposite as its natural orientation; therefore

$$\partial V = \sum_{i=1}^{n} B_{p_i}.$$

Where $\sum$ denotes the formal sum. It also can be understood as the disjoint union among all the B_{p_i}. Then

$$\int_V \mathrm{K}\, dA = -\int_V d\omega_j^i.$$

Using Stokes's theorem

$$\begin{aligned} \int_V \mathrm{K}\, dA &= -\int_{\partial V} \omega_j^i \\ &= -\sum_{i=1}^{n} \int_{\partial B_{p_i}} \omega_2^1. \end{aligned}$$

When r_i goes to 0 and using lemma (27) we have

$$\int_M \mathrm{K}\, dA = 2\pi \sum_{i=1}^{n} \mathrm{Ind}_X(p_i).$$

$\square$

Theorem 29. (**Poincaré–Hopf**) Let M be a smooth compact manifold with boundary, and let X be a vector field on M with isolated singularities such that X points outwards at all points in the boundary. Then

$$\sum_{i=1}^{n} \operatorname{Ind}_X(p_i) = \chi(M).$$

Where $\{p_i\}_{i=1}^n$ is the set of the singularities of X, and $\chi(M)$ is the well-known *Euler-Poincaré* characteristic.

This theorem provides us the classic version of 3.20 as it is stated bellow.

Theorem 30. (**Gauß–Bonnet theorem**) For every compact oriented 2-dimensional manifold, we have

$$\int_M \mathrm{K}\, dA = 2\pi\chi(M).$$

3.5 Gauß–Bonnet–Chern

We start by defining the quantities needed to generalize the Gauß–Bonnet Theorem 3.20.

Theorem 31. For a vector bundle the connection form $\omega = (\omega^i_j)$ and the curvature form $\Omega = (\Omega^i_j)$ are related by

$$\Omega = d\omega + \omega \wedge \omega.$$

Proof. From the definition

$$\begin{aligned}
\mathrm{R}(X,Y)s_j &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})s_j \\
&= \nabla_X(\sum_{i=1}^{n}\omega^i_j(Y)s_i) - \nabla_Y(\sum_{i=1}^{n}\omega^i_j(X)s_i) - \sum_{i=1}^{n}\omega^i_j([X,Y])_i \\
&= \sum_{i=1}^{n} X\omega^i_j(Y)s_i + \sum_{i=1}^{n}\omega^k_j(Y)\omega^i_k(X)s_i - \sum_{i=1}^{n} Y\omega^i_j(X)s_i - \sum_{i=1}^{n}\omega^k_j(X)\omega^i_k(Y)s_i \\
&- \sum_{i=1}^{n}\omega^i_j([X,Y])_i
\end{aligned}$$

Substitute $d\omega^i_j(X,Y) = X\omega^i_j(Y) - Y\omega^i_j(X) - \omega^i_j([X,Y])$ and $\omega^i_k \wedge \omega^k_j(X,Y) = \omega^i_k(X)\omega^k_j(Y) - \omega^i_k(Y)\omega^k_j(X)$ to get

$$= \mathrm{R}(X,Y)s_j = \sum_{i=1}^{n} d\omega^i_j(X,Y) + \sum_{k=1}^{n}\omega^i_k \wedge \omega^k_j(X,Y)s_j$$

Finally, from the definition of the curvature form we get the desired result

$$\mathrm{R}(X,Y)s_j = \sum_{i=1}^{n} \Omega_j^i(X,Y)s_j.$$

$\square$

Euler Class

Let E be a vector bundle on M. when we select a Riemannian metric on E and a connection, the curvature form Ω is represented by a skew-symmetric matrix. Moreover, if we take a skew-symmetric matrix, then its determinant satisfies, in particular, two interesting properties. If X is an $n \times n$ skew-symmetric matrix, denoted by $X \in \mathrm{Skew}_n$; then if n is odd, its determinant vanishes; on the other hand, when n is even, the determinant of X can be written as the squared of a polynomial. In this case, we define the **Pfaffian** of X as $\det(X) = (\mathrm{Pf}(X))^2$, which is

$$\mathrm{Pf}(X) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} (-1)^{\mathrm{sgn}(\sigma)} x_{\sigma_2}^{\sigma_1} \cdots x_{\sigma_{2n}}^{\sigma_{2n-1}}.$$

In our particular case, we can construct a $2n$-form on M by setting

$$\mathrm{Pf}(\Omega) \in \mathcal{A}^{2n}(M)$$

Since the Pfaffian has an invariance property under orthogonal transformations, i.e. if $T \in O(2n)$, then $\mathrm{Pf}(T^{-1}\Omega T) = \det(T)\,\mathrm{Pf}(\Omega)$. Note that when $T \in SO(2n)$, we have that $\det(T) = 1$. Furthermore, for the corresponding curvature form Ω we have

$$\mathrm{Pf}(\Omega) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} (-1)^{\sigma} \Omega_{\sigma_2}^{\sigma_1} \cdots \Omega_{2n}^{2n-1}$$

Then we define the *Euler form* by setting

$$\mathrm{eu}(\Omega) = \frac{1}{(2\pi)^n}\,\mathrm{Pf}(\Omega) = \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right).$$

There is an important property that satisfies the Euler form. It is stated next.

Theorem 32. (**Gauß-Bonnet theorem, general version**) Let M be an oriented $2n$-dimensional compact manifold, then we have

$$\chi(M) = \int_M \mathrm{eu}(\Omega).$$

The proof of this theorem can be written in some different ways. As we want to take a look at the geometry acting behind the theorem, we will provide a proof given by Morita S. in [14] in which the vector field is written explicitly.

Lemma 33. Let E be a vector bundle, if there exists a section on E that is never 0, then $\mathrm{e}(E) = 0$. Here e denotes the *Euler class* given by $[\mathrm{eu}(\Omega)] \in H^{2n}(M; \mathbb{R})$.

Proof. Let $s(p)$ be a section on E such that s in never 0; assume $|s| = 1$. It is possible to construct a connection ∇ such that $\nabla s = 0$. Let $\{s_i\}$ be a frame field such that $s_1 = s$, thus the corresponding connection form has its first column equals to 0. From the structure equation, the curvature form has the same columns identically equals to 0. For that, $\mathrm{Pf}(\Omega) = 0$ and thus the Euler form vanishes too. $\square$

Sketch of the proof of the Gauß–Bonnet theorem. We shall view first the main steps of the proof.

1. It is possible to made a vector field on M via triangulations such that there is a singularity in the barycenter.

2. We shall show that $\mathrm{eu}(\Omega)$ is 0 except in a neighborhood of each singularity.

3. Then, it is time to prove $\int_M \mathrm{eu}(\Omega) = \sum_\sigma (-1)^{\dim \sigma} = \chi(M)$.

$\square$

Proof of the Gauß-Bonnet theorem. The first step is to construct the vector field whose barycenter is a singularity. Let $t : K \to M$ be a triangulation of M. Let $K^{'}$ be the complex obtained by barycenter subdivision of K, and let $K^{''}$ be the complex obtained by applying the procedure once again. Then we define a simplicial map from $K^{''}$ to $K^{'}$ by taking any vertex $v \in V(K^{''})$, with $V(K^{''})$ the set of all vertices of $K^{''}$, and sending it into the barycenter of a simplex, this simplex is the only one in K where v is contained; i.e.

$$\varphi : V(K^{''}) \subset K^{''} \to V(K^{'}) \subset K^{'},$$
$$\varphi(v) = b_v.$$

Where b_v is the barycenter of the simplex in K. This fact is illustrated in the following figures.

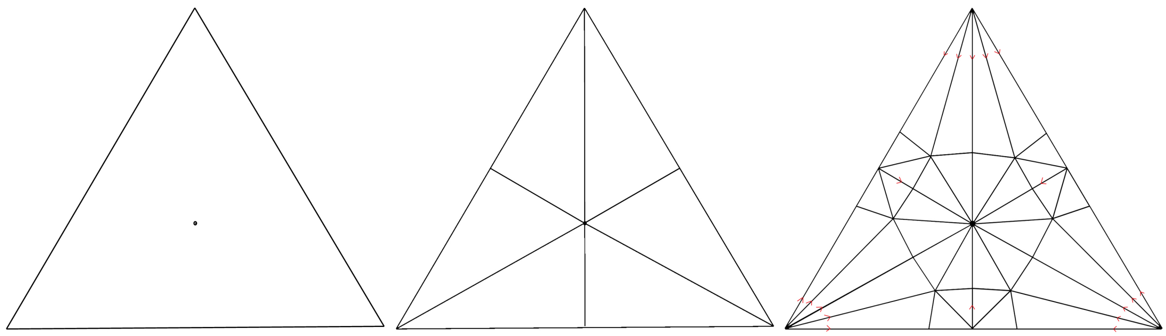


Figure 3.1: K, $K^{'}$, $K^{''}$, respectively.

Note that the set of all fixed points by φ is the set of barycenters of simplices of K, that is to say, $V(K^{'})$. After this, we remark that for each $p \in M$, the segment that joins p with $\varphi(p)$ inside the simplex of K has an image by the map t, it is actually a curve $t(p)$. Then, let X_p be the vector $\dot{t}(p)$ representing the velocity at $t(p)$, and then we reparametrize by arclength, so as p approaches a singularity, the length of X_p goes to 0. This vector field is continuous; however, it is not smooth. For this, we need the fact that every continuous map can be approximated by C^∞ objects. Let X be the smoothen vector field, and we now analise how X acts at each singularity b_σ. Let $\sigma \in K$ be a simplex with dimension i. If $i = 0$ and q is a vertex of K, then X diverges away from q. On the other hand, if $i = 2n$, then q is a barycenter of a simplex of K of highest dimension and X converges to q. To sum up, if $0 < i < 2n$, X converges in i directions tangent to σ and diverges toward the barycenters of the $2n$-dimensional simplexes in the other $(2n - i)$ dimensions. Hence, our smoothen vector field with isolated singularity q has the form

$$Y_i = -(x_1 \frac{\partial}{\partial x_1} + \cdots + x_i \frac{\partial}{\partial x_i}) + x_{i+1} \frac{\partial}{\partial x_{i+1}} + \cdots + x_{2n} \frac{\partial}{\partial x_{2n}}. \tag{3.21}$$

Then we showed using the previous lemma that if we take a good connection in TM, then $\mathrm{eu}(\Omega)$ vanishes except in a neighborhood U_q of $q \in V(K^{'})$, and moreover we have,

$$\int_{U_q} \mathrm{eu}(\Omega) = (-1)^{i_q}, \tag{3.22}$$

where i_q is the dimension of the simplex of K having q as its barycenter. Indeed, $(-1)^{i_q} = \mathrm{Ind}_X(q)$, so we get

$$\int_M \mathrm{eu}(\Omega) = \sum_{q \in V(K^{'})} \int_{U_q} \mathrm{eu}(\Omega) = \sum_{q \in V(K^{'})} (-1)^{i_q} = \sum_q \mathrm{Ind}_X(q) = \chi(M).$$

This complete the proof. The only detail that remains to prove is equation 3.22. Let us see what happen in the two dimensional case. Here, there are three possible vector fields, Y_0, Y_1, and Y_2. Let Y_0' be the normalized first vector field. We will construct a connection in $T\mathbb{R}^2$ such that $\nabla Y_0' = 0$ out of each singularity. Then Y_0' is

$$Y_0' = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}.$$

Consider the connection form $\omega = (\omega^i_j)_{ij}$ relative to $\frac{\partial}{\partial x}$, and $\frac{\partial}{\partial y}$. As $\omega^1_2 = -\omega^2_1$ we get

$$\nabla\frac{\partial}{\partial x} = -\omega^1_2 \otimes \frac{\partial}{\partial y}, \quad \nabla\frac{\partial}{\partial y} = \omega^1_2 \otimes \frac{\partial}{\partial x}.$$

Then we get

$$\begin{aligned} Y_0' &= d\left(\frac{x}{\sqrt{x^2+y^2}}\right)\otimes\frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}}\nabla\frac{\partial}{\partial x} + d\left(\frac{y}{\sqrt{x^2+y^2}}\right)\otimes\frac{\partial}{\partial y} + \frac{y}{\sqrt{x^2+y^2}}\nabla\frac{\partial}{\partial y} \\ &= \left(\frac{y^2\,dx - xy\,dy}{\sqrt{(x^2+y^2)^3}} + \frac{y}{\sqrt{x^2+y^2}}\omega^1_2\right)\otimes\frac{\partial}{\partial x} + \left(\frac{x^2\,dy - xy\,dx}{\sqrt{(x^2+y^2)^3}} - \frac{x}{\sqrt{x^2+y^2}}\omega^1_2\right)\otimes\frac{\partial}{\partial y}. \end{aligned}$$

Then the connection form is

$$\omega^1_2 = \frac{-y}{x^2+y^2}dy + \frac{x}{x^2+y^2}dx.$$

With $\tan(\theta) = {}^{y}\!/_{x}$ we get

$$\omega^1_2 = d\theta$$

The structure equations show that

$$\Omega^1_2 = d\omega^1_2.$$

Hence $\int_{D(r)} \Omega^1_2 = \int_{S^1} d\theta = 2\pi$. Therefore

$$\int_{D(r)} \mathrm{eu}(\Omega) = \int_{D(1)} \tfrac{1}{2\pi}\,\mathrm{Pf}(\Omega) = \int_{D(1)} \tfrac{1}{2\pi}\Omega^1_2 = 1.$$

Then 3.22 has sense. For Y_1, and Y_2 it is analogous obtaining -1 and 1, respectively. In general, let $Y_i^{'}$ be the normalized vector field given by $\frac{Y_i}{\|Y_i\|}$. Let ∇ be a connection on $T\mathbb{R}^{2n}$ such that $\nabla Y_i^{'} = 0$ away from the singularity. We will show that the value of the integral

$$\int_{D(1)} \mathrm{eu}(\Omega) = a_{n,i}$$

is $(-1)^i$. then for any n, we can write $a_{n,i} = (-1)^i a_{n,0}$ as we have seen in the 2-dimensional case. Hence it is enough to prove that $a_{n,0} = 1$. It is defined a vector field on S^2 from the north pole to the south pole, so Z only has its two singularities at those points. Since $a_{n,i}$ does not depend on the manifold we take $M_n = S^2 \times \cdots S_2$ as the direct product of n spheres S^2, and let $\pi_i : M_n \to S^2$ be the projection onto its i-th coordinate. Then we have

$$TM_n \cong \pi^* TS^2 \oplus \cdots \oplus \pi^* TS^2.$$

Thus, it is possible to define a vector field Z_n by

$$Z_n(p_1, \ldots, p_n) = Z(p_1) \otimes \cdots \otimes Z(p_n).$$

Therefore Z_n has singularities at the poles, there are 2^n of them in total. Therefore

$$\int_{M_n} \mathrm{eu}(\Omega) = 2^n a_{n,0}.$$

On the other hand

$$[\mathrm{eu}(TM_n)] = \pi_1^*[\mathrm{eu}(TS^2)] \cdots \pi_1^*[\mathrm{eu}(TS^2)],$$

then

$$\int_{M_n} \mathrm{eu}(\Omega) = \left(\int_{S^2} \mathrm{eu}(\Omega) \right)^n = 2^n.$$

Which complete the proof. $\square$

Chapter 4

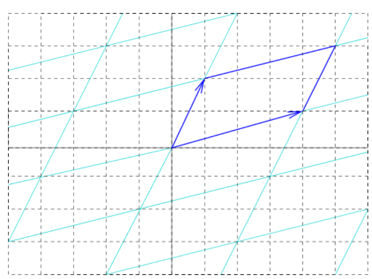
Examples

Definition 34. Let Λ be a lattice on $\mathbb{R}^n$ given by

$$\Lambda^n = \left\{ \sum_{i=1}^{n} m_k e_k \mid m_k \in \mathbb{Z} \right\}.$$

Then any flat torus of dimension n has the form $\mathbb{T}^n = \mathbb{R}^n / \Lambda^n$.

Example 35. All flat tori of any dimension have curvature equal to 0, for this $\mathrm{eu}(\Omega) = 0$ and its integral vanishes.



Example 36. This first example illustrates how useful the theorem 3.1 is. So, we shall work on both sides of the equality over the embedded torus $\mathbb{T}^2 \in \mathbb{R}^3$with the induced metric. Consider the following parametrization.

$$T(\theta, \varphi) = \begin{pmatrix} (R + r\cos\theta)\cos\varphi \\ (R + r\cos\theta)\sin\varphi \\ r\sin\varphi \end{pmatrix},$$

where θ, and φ run over the interval $[0, 2\pi]$. Since the torus $\mathbb{T}^2$ can be seen as a surface of revolution, then we calculate the components of the first fundamental form (metric) $I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

by the following products.

$$I_p = \begin{pmatrix} \langle T_\theta, T_\theta \rangle & \langle T_\theta, T_\varphi \rangle \\ \langle T_\varphi, T_\theta \rangle & \langle T_\varphi, T_\varphi \rangle \end{pmatrix}.$$

So with $T_\theta = (-r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi, r\cos\theta)$ and $T_\varphi = (-(R+r\cos\theta)\sin\varphi, (R+r\cos\theta)\cos\varphi, 0)$, we get the first fundamental form of T^2,

$$I_p = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos\theta)^2 \end{pmatrix}. \tag{4.1}$$

Recall that the Gaussian map (or normal map) $\widehat{N}$ is defined by $\tilde{N} = \frac{T_\theta \times T_\varphi}{|T_\theta \times T_\varphi|}$ and equal to

$$\widehat{N}(\theta, \varphi) = -(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta).$$

Next, we need to find the second derivatives $T_{\theta\theta}$, $T_{\theta\varphi}$, $T_{\varphi\theta}$ and $T_{\varphi\varphi}$ to calculate the second fundamental form $II_p = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ where $e = \langle \widehat{N}, T_{\theta\theta} \rangle$, $f = \langle \widehat{N}, T_{\theta\varphi} \rangle$ and $g = \langle \widehat{N}, T_{\varphi\varphi} \rangle$, then

$$\begin{aligned} T_{\theta\theta} &= (-r\cos\theta\cos\varphi, -r\cos\theta\sin\varphi, -r\sin\theta), \\ T_{\theta\varphi} &= (r\sin\theta\sin\varphi, -r\sin\theta\cos\varphi, 0), \\ T_{\varphi\theta} &= (r\sin\theta\sin\varphi, -r\sin\theta\cos\varphi, 0), \\ T_{\varphi\varphi} &= (-(R + r\cos\theta)\cos\varphi, -(R + r\cos\theta)\sin\varphi, 0), \end{aligned}$$

and finally

$$II_p = \begin{pmatrix} r & 0 \\ 0 & (R + r\cos\theta)\cos\theta \end{pmatrix}.$$

Hence the Weingarten operator S_p, defined by $II_p \cdot I_p^{-1}$, is

$$S_p = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{\cos(\theta)}{(R+r\cos(\theta))} \end{pmatrix}.$$

thus,

$$\mathrm{K} = \det(S_p) = \frac{\cos(\theta)}{r(R + r\cos(\theta))}.$$

Furthermore,

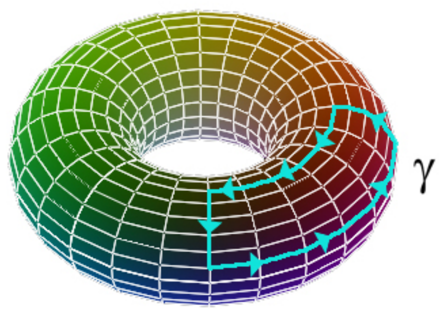
$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos(\theta)}{r(R + r\cos(\theta)} dA = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(\theta) d\theta d\varphi \ = \frac{\pi}{2}$$

Since $dA = \sqrt{EG - F^2} d\theta\, d\varphi$, and $E = r^2$, $G = (R + r\cos(\theta))^2$, and $F = 0$. On the other hand, with γ as follows we shall integrate $\int_0^L \kappa_g(s)ds$ where the geodesic curvature κ_g is defined by

$\kappa_g = \langle \widehat{N}(\gamma(s)) \times \dot{\gamma(s)}, \ddot{\gamma(s)} \rangle,$

$$\gamma(t) = \begin{cases} (t\frac{\pi}{2}, 0) & t \in [0,1] \\ (\frac{\pi}{2}, t\frac{\pi}{2}) & t \in [0,1] \\ (\frac{\pi}{2} - t\frac{\pi}{2}, \frac{\pi}{2}) & t \in [0,1] \\ (0, \frac{\pi}{2} - t\frac{\pi}{2}) & t \in [0,1] \end{cases}$$

This fact is illustrated in the following image.



Recall that here it is important to keep in mind that γ has to be parametrized by acrlength, for that we use the induced metric $(\frac{ds}{dt})^2 = G(\frac{d\varphi}{dt})^2 + 2F(d\varphi\, d\theta) + E(\frac{d\theta}{dt})^2$ to find the parameter s as a function of t. For instance, for the second piece we have

$$\frac{ds}{dt}^2 = (R + r\cos\theta)^2 \frac{d\varphi}{dt}^2 + r^2 \frac{d\theta}{dt}^2$$

In this case, $\varphi = \frac{\pi}{2}t$, $\frac{d\varphi}{dt} = \frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$, $\frac{d\theta}{dt} = 0$, so we get

$$\begin{aligned} \frac{ds}{dt}^2 &= (R + r\cos(\frac{\pi}{2}))^2 \frac{d\varphi}{dt}^2 \\ \frac{ds}{dt} &= R\frac{d\varphi}{dt} = \frac{\pi}{2}R \\ s &= \frac{\pi\, R}{2}t \end{aligned}$$

Next, after calculating γ along the normal map $\widehat{N} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \cos\theta)$ with $(\frac{\pi}{2}, \frac{\pi}{2}t) = (\frac{\pi}{2}, \frac{s}{R})$, and of course, $\dot{\gamma}$ and $\ddot{\gamma}$,

$$\widehat{N}(\gamma_2(s)) = (0, 0, 1).$$

Where γ_2 denotes the second piece of γ,

$$\begin{aligned} T(\dot{\gamma}_2(s)) &= (-\sin(s/R), \cos(s/R), 0) \\ T(\ddot{\gamma}_2(s)) &= (-\frac{1}{R}\cos(s/R), -\frac{1}{R}\sin(s/R), 0) \end{aligned}$$

the geodesic curvature of γ is:

$$\kappa_g(s) = \frac{1}{R}\cos^2\left(\frac{s}{R}\right) + \frac{1}{R}\sin^2\left(\frac{s}{R}\right) = \frac{1}{R}.$$

Then the integral of the geodesic curvature is

$$\int_0^{\frac{\pi R}{2}} \frac{1}{R} ds = \frac{\pi}{2}.$$

After calculating κ_g of the other paths by analogy, we find that all γ_1 , γ_3, and γ_4 has geodesic curvature identically 0 since they are parallels and meridians.

Example 37. Now we will take a look at all the surfaces in $\mathbb{R}^3$ by linking toruses properly. Since every 2-dimensional compact manifold in $\mathbb{R}^3$ is homomorphic to a 2-dimensional manifold that coincides with its number of holes, we can join as many tori as needed. This property is illustrated in the next graphic.

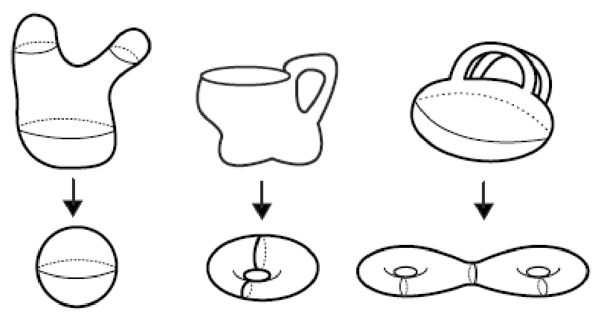
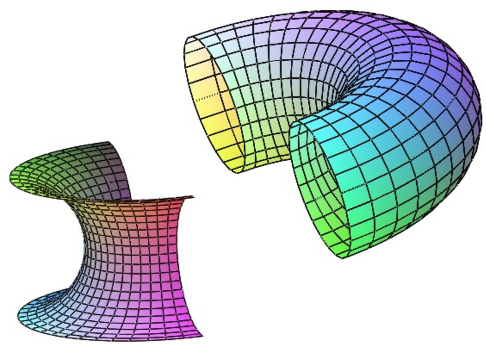


Figure 4.1: Available online on: http://www.learner.org

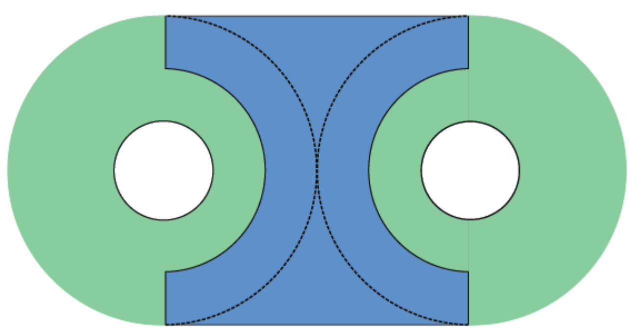
Using example 36 we are able to calculate $\int_M \mathrm{K}\, dA$ by adding pieces of tori as long as the angle runs a length of $\frac{\pi}{2}$. The following construction illustrates how this gluing is made. Using the parametrization of the torus

$$T(\theta,\varphi) = \begin{pmatrix} (R + r\cos\theta)\cos\varphi \\ (R + r\cos\theta)\sin\varphi \\ r\sin\varphi \end{pmatrix}.$$

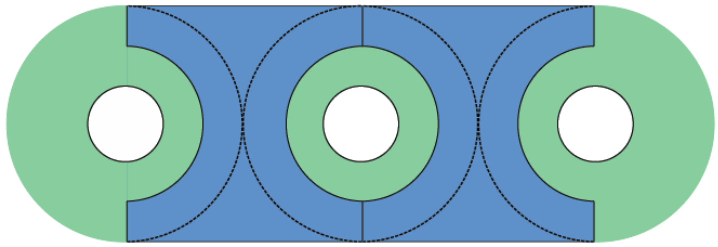
Where θ and φ runs in the interval $[0, 2\pi]$. We have now to think of the torus as two halves, one when $\theta \in [\pi/2, 3\pi/2]$ and $\varphi \in [0, 2\pi]$ and the other one when $\theta \in [0, 2\pi]$ and $\varphi \in [\pi, 2\pi]$. Note here that the left part indeed has $\int_R \mathrm{K}\, dA = -\pi$, R the integrating region. This fact is showed in the next plot.



After this, glue this piece of torus with other piece of torus divided in the same way through a cylindrical surface S whose curvature vanishes at all points $p \in S$. Concretely, with two planes at the top and the bottom, and one half of a circular cylinder at each side. A top view is showed next.



Similarly, we can add as many holes as we want to.



So, for each of the two tori at the ends we add -2π and for each tori in the middle we add -4π. This provides a different and easy way to calculate $\int_R \mathrm{K}\, dA$. For instance, take the previous 3-holes surface, then the green part in the middle has an integral of its curvature equal to -4π and at each of the ends it sums up -2π. Thus in this surface $2\pi\chi(M) = -8\pi$.

Example 38. This example uses the equation 1.1, and of course the Gauß–Bonnet theorem in high dimensions. Consider the product $T^{2n} = \underbrace{T^2 \times \cdots \times T^2}_{n\text{-times}}$. For T^2 we have $\omega_2^1 = \langle \mathrm{R}(e_1, e_2)e_1, e_2 \rangle = K = \frac{\cos\theta}{r(R+r\cos\theta)}$ then the integral of the Pfaffian over T^{2n} is the n-product of the integrals of the Pfaffian of T^2, i.e.

$$\begin{aligned}
\int_{T^{2n}} \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right) &= \left(\frac{1}{2\pi}\int_{T^2} \mathrm{Pf}(\Omega)\right)^n \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos\theta}{r(R+r\cos\theta)} dA\right)^n \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \cos\theta\, d\theta\, d\varphi\right)^n \\
&= 0 \\
&= \chi(T^{2n})
\end{aligned}$$

Example 39. Now, we will look at the product of n spheres S^2 of the form $S^2 = \{x \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ whose curvature is 1. Then as in the previous example we get

$$\begin{aligned}
\int_{s^{2n}} \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right) &= \left(\frac{1}{2\pi}\int_{s^2} \mathrm{Pf}(\Omega)\right)^n \\
&= \left(\frac{1}{2\pi}\int_0^{\pi}\int_0^{2\pi} \sin\varphi\, d\theta\, d\varphi\right)^n \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi\right)^n \\
&= ([-\cos\varphi]_0^{\pi})^n \\
&= 2^n \\
&= \chi(S^2)^n \\
&= \chi(S^{2n}).
\end{aligned}$$

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