The Riemann Zeta Function

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Contents

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Introduction

In the eighteen century, the swiss mathematician Leonhard Euler (1707–1783) introduced the zeta function defined for $s \in \mathbb{R}$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In elementary courses of calculus, one of the first examples of an infinite series is the one given by $\zeta(s)$. Using the integral test the student learns that

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for $s > 1$ and diverges if $s \leqslant 1$. Some enthusiastic teachers will point out the fact that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is known as the *Basel problem*, and it was solved by Euler in 1734. Euler's proof depended on some assumptions that are rather difficult to justify. For example, at a key point in the solution, Euler observed that the function

$$\frac{\sin x}{x}$$

and the infinite product

$$\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{2\pi}\right)\cdots \tag{0.0.1}$$

have exactly the same roots and have the same value at $x = 0$, so Euler asserted that they describe the same function. As we shall see (Corollary 3.4.1.1) Euler was right but the reasons just mentioned are insufficient to guarantee it. For example, the function

$$\mathrm{e}^x \frac{\sin x}{x}$$

also has the same roots and the same value at $x = 0$, but it is a different function. It was not until 100 years later, that Karl Weirstrass proved that Euler's representation of the sine function as an infinite product was valid, by the Weirstrass factorization theorem. Euler's approach to the Basel problem goes as follows: If we formally multiply the product in (0.0.1) we find out that the coefficient of x^2 is

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right) = -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

But from the Taylor expansion of $\sin x$ around $x = 0$ we know that

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots,$$

and hence the coefficient of x^2 is also $\frac{1}{6}$. This shows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Euler used the same approach to find an expression for $\zeta(2m)$ for $m \in \mathbb{N}$, namely

$$\zeta(2m) = \frac{(2\pi)^{2m}(-1)^{n+1}B_{2m}}{2(2m)!}, \qquad m = 1, 2, \ldots,$$

where B_{2m} are the Bernoulli numbers which will be introduced in Section 1.3. This result not only provides an elegant formula for evaluating $\zeta(2m)$, but it also tells us the arithmetic nature of $\zeta(2m)$. In contrast, we know very little about the odd zeta values $\zeta(2m+1)$. It is known that $\zeta(3)$, the *Apéry's constant*, is irrational (Apéry 1978) and that there are infinitely many $m \in \mathbb{N}$ such that $\zeta(2m+1)$ is irrational (Rivoal 2000); for a proof of these facts see [5]. Even so, we don't know the nature of $\zeta(2m+1)$; for instance we don't know whether $\zeta(5)$ is irrational or not.

After Euler achieved his objective of evaluating $\zeta(2)$, he then turned to the arithmetic properties of $\zeta(s)$. In 1737 he published a paper entitled *Variae observationes circe series infinitas* (Various observations about infinite series). Here for the first time he proved the famous Euler product formula in the form

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}} \qquad (s > 1), \tag{0.0.2}$$

where $\mathbb{P}$ denotes the set of prime numbers. This result is fascinating because it shows that the zeta function is related to the prime numbers. Euler then used this result to prove that the series

$$\sum_{p \in \mathbb{P}} \frac{1}{p}$$

diverges. Euler's proof goes as follows: by taking logarithms to each side of (0.0.2) we obtain

$$\log \zeta(s) = -\sum_{p \in \mathbb{P}} \log(1 - p^{-s}).$$

Since

$$-\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad (|x| < 1),$$

and $|p^{-s}| < 1$, then

$$\begin{aligned} \log \zeta(s) &= \sum_{p \in \mathbb{P}} \frac{1}{p^s} + \sum_{p \in \mathbb{P}} \frac{1}{p^{2s}} \left(\frac{1}{2} + \frac{1}{3p^s} + \frac{1}{4p^{2s}} + \cdots \right) \\ &< \sum_{p \in \mathbb{P}} \frac{1}{p^s} + \sum_{p \in \mathbb{P}} \frac{1}{p^s(p^s - 1)} \\ &< \sum_{p \in \mathbb{P}} \frac{1}{p^s} + \sum_{n=1}^{\infty} \frac{1}{n^{2s}}. \end{aligned}$$

Since the harmonic series diverges, letting $s \to 1$ we conclude that

$$\sum_{p \in \mathbb{P}} \frac{1}{p}$$

diverges. In 1740 Euler published a paper entitled *De Serbius Quibusdam Considerationes*. In this paper he computed approximate values of $\zeta(2m+1)$ for $m = 1, 2, 3, 4, 5$ to which he added the known values of $\zeta(2m)$. He wrote this in the form

$$\zeta(n) = N\pi^n,$$

and said that if n is even, then N is rational, while if n is odd he conjectures that N is a function of $\log 2$. In the middle of his paper, Euler states that

$$\begin{aligned} 1 - 3 + 5 - 7 + \cdots &= 0, \\ 1 - 3^3 + 5^3 - 7^3 + \cdots &= 0, \end{aligned}$$

and so on, while

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \log 2, \\ 1 - 2 + 3 - 4 + \cdots &= \frac{1}{4}, \\ 1 - 2^3 + 3^3 - 4^3 + \cdots &= -\frac{1}{8}. \end{aligned}$$

On the other hand,

$$\begin{aligned} 1 - 2^2 + 3^2 - 4^2 + \cdots &= 0, \\ 1 - 2^4 + 3^4 - 4^4 + \cdots &= 0, \\ 1 - 2^6 + 3^6 - 4^6 + \cdots &= 0. \end{aligned}$$

Euler derived this identities as follows: Let

$$f(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x} \qquad (|x| < 1).$$

Euler had no reluctance to let $x = -1$; then

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}.$$

Now he applied the operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ to $f(x)$ and obtained

$$x\frac{\mathrm{d}}{\mathrm{d}x} f(x) = x + 2x^2 + 3x^3 + \cdots = \frac{1}{(1-x)^2};$$

then, he let $x = -1$ to obtain

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}.$$

Applying the operator again and evaluating at $x = -1$ gives

$$1 - 2^2 + 3^2 - 4^2 + \cdots = 0.$$

Since the series converges at each stage of this process for $|x| < 1$, we see that Euler anticipated Abel summability[1] by some 75 years.

In 1749 he gave a paper to the Berlin Academy entitled *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques*(Remarks on a beautiful relation between direct as well as reciprocal power series). In his papers he considers the function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},$$

and using the methods described before, and the formulas for $\zeta(2m)$ he notes that

$$\begin{aligned}
\frac{1-2+3-4+5-6+\cdots}{1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots} &= \frac{1\cdot(2^2-1)}{(2-1)\pi^2},\\
\frac{1^2-2^2+3^2-4^2+5^2-6^2+\cdots}{1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{6^3}+\cdots} &= 0,\\
\frac{1^4-2^3+3^3-4^3+5^3-6^3+\cdots}{1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\cdots} &= -\frac{1\cdot 2\cdot 3\cdot(2^4-1)}{(2^3-1)\pi^4},\\
\frac{1^4-2^4+3^4-4^4+5^4-6^4+\cdots}{1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\frac{1}{6^5}+\cdots} &= 0,
\end{aligned}$$

or if $m \geqslant 2$,

$$\frac{\eta(1-m)}{\eta(m)} = \begin{cases} \frac{(-1)^{\frac{m}{2}+1}(2^m-1)(m-1)!}{(2^{m-1}-1)\pi^m}, & \text{if } m \text{ is even;} \\ 0, & \text{otherwise.} \end{cases} \tag{0.0.3}$$

Euler listed these relations for $m = 2, 3, \cdots, 10$. On the other hand if $m = 1$, we see that

$$\frac{1-1+1-1+\cdots}{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots} = \frac{1}{2\log 2}.$$

Then Euler wrote (0.0.3) in the form

$$\frac{\eta(1-m)}{\eta(m)} = -\frac{(m-1)!(2^m-1)\cos\frac{\pi m}{2}}{(2^{m-1}-1)\pi^m},$$

and then he said "I shall hazard the following conjecture:

$$\frac{\eta(1-s)}{\eta(s)} = -\frac{\Gamma(s)(2^s-1)\cos\frac{\pi s}{2}}{(2^{s-1}-1)\pi^s} \tag{0.0.4}$$

is true for all s". Here Γ is the Gamma function which will be discussed in Section 2.1. Then, Euler continue saying: "The validity of our conjecture for $s = 1$ (which appeared to be deviated from the others) is already a strong justification to our conjecture, since it appears unlikely that a false assumption could support this case. Therefore, we can regard our conjecture as being solidly based but I shall give other justifications which are equally convincing". Euler then proceeds to check the formula for $s = \dfrac{2k+1}{2}$. Note that

$$\eta(s) = (1-2^{1-s})\zeta(s).$$

[1]Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers. We say that $\sum_{n\in\mathbb{N}} a_n$ is Abel summable if $\lim_{x\to 1^-}\sum_{n\in\mathbb{N}} a_n x^n$ exists and is finite. For example $1-2+3-4+\cdots$ is Abel summable since $x+2x^2+3x^3+\cdots$ converges for every $|x| < 1$.

Therefore, using (0.0.4) we can write

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{s\pi}{2}\right)\Gamma(s)\zeta(s),$$

and this is the unsymmetrical form of the famous functional equation proved by Riemann in 1859. The function $\zeta(s)$ defined by Euler is nowadays called the *Riemann zeta function* because the german mathematician Bernhard Riemann (1826–1866) was the first to study extensively its properties. In 1859, he wrote a short paper (8 pages) called *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the number of primes less than a given magnitude) in which he expressed fundamental properties of $\zeta(s)$ in the complex variable $s = \sigma + it$. We state these in the modern nomenclature.

1. The Riemann zeta function $\zeta(s)$ extends to a meromorphic function on the whole complex plane with a simple pole at $s = 1$, and the function

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

satisfies the functional equation

$$\xi(s) = \xi(1-s)$$

for every $s \in \mathbb{C}$.

2. $\zeta(s)$ has simple real zeros at $s = -2, -4, -6, \ldots$, which are called the *trivial zeros*, and infinitely many non-trivial zeros of the form

$$\rho = \beta + i\gamma, \qquad 0 \leqslant \beta \leqslant 1, \gamma \in \mathbb{R}.$$

The number $N(T)$ of non-trivial zeros of height $0 < \gamma < T$ satisfies

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T).$$

This was proved by von Mangoldt in 1905.

3. The entire function ξ has the following product representation:

$$\xi(s) = \mathrm{e}^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)\mathrm{e}^{\frac{s}{\rho}},$$

where A and B are constants and ρ runs through the non trivial zeros of ζ in the critical strip. This was proved by Hadamard in 1893. It played an important role in the proofs of the prime number theorem by Hadamard and de la Vallée Poussin.

4. For $n \in \mathbb{N}$ we define the *von Mangoldt function* as

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some } p \in \mathbb{P} \text{ and for some } m \in \mathbb{N}^*; \\ 0, & \text{otherwise,} \end{cases} \tag{0.0.5}$$

and the *Chebyshev's ψ function* as

$$\psi(x) = \sum_{n \leqslant x}\Lambda(n).$$

Then for $x > 1$ not a prime power we have

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right),$$

where the sum over the nontrivial zeros $\rho = \beta + i\gamma$ of ζ is to be understood in the symmetric sense as

$$\lim_{T\to\infty} \sum_{|\gamma|\leqslant T} \frac{x^\rho}{\rho}.$$

This was proved by von Mangoldt in 1895.

5. The Riemann hypothesis, one of the most important unsolved problems in mathematics:

   Conjecture (The Riemann Hypothesis)**.** *Every non trivial zero of $\zeta(s)$ is on the line $\mathfrak{Re}(s) = \frac{1}{2}$.*

The goal of this document is to provide a formal proof of all the conjectures and theorems made by Euler and Riemann (except for the Riemann hypothesis).

Chapter 1

Preliminaries

1.1 Some Theorems from Analysis

When manipulating series and integrals we often encounter problems related with interchanging summation and integration, and differentiation of integrals with respect to a parameter. In this section we quote some tools that are frequently used in analysis.

The first theorem gives the condition to justify what is usually named *interchanging summation and integration*. It is called the *theorem of dominated convergence of Lebesgue* in the setting of Riemann integrals. A proof can be found for example in [1].

Theorem 1.1.1 (Dominated convergence theorem of Lebesgue)**.** *Let $\{f_n(t)\}_{n\in\mathbb{N}}$ be a sequence of complex-valued functions which are continuous in $(a,b)\subseteq\mathbb{R}$ and have the properties*

(i) $\sum_{n=1}^{\infty} f_n(t)$ converges uniformly in any compact interval in (a,b).

(ii) At least one of the following quantities is finite:

$$\int_a^b \sum_{n=1}^{\infty} |f_n(t)|\,\mathrm{d}t, \qquad \sum_{n=1}^{\infty}\int_a^b |f_n(t)|\,\mathrm{d}t.$$

Then

$$\int_a^b \sum_{n=1}^{\infty} f_n(t)\,\mathrm{d}t = \sum_{n=1}^{\infty}\int_a^b f_n(t)\,\mathrm{d}t.$$

The second theorem is the general version of partial integration, the proof runs by induction on m and the details are left to the reader.

Theorem 1.1.2 (Integration by parts)**.** *Let $h\in C(\alpha,\beta)$ and $g\in C^m(\alpha,\beta)$. Then, for every $m\in\mathbb{N}$*

$$\int_\alpha^\beta g(x)h(x)\,\mathrm{d}x = \sum_{k=0}^{m-1}(-1)^k g^k(x)h^{-(k+1)}(x)\Big|_\alpha^\beta + (-1)^m\int_\alpha^\beta g^m(x)h^{-m}(x)\,\mathrm{d}x,$$

where h^{-m} denotes the mth integral of h.

The third theorem is an extension to complex variables of a standard theorem concerning differentiation of an integral over an infinite contour with respect to a parameter; for a proof see for example [3].

Theorem 1.1.3. *Let t be a real variable ranging over a finite or infinite interval (a, b) and z a complex variable ranging over a domain Ω. Assume that the function $f\colon \Omega \times (a, b) \to \mathbb{C}$ satisfies the following conditions:*

(i) *f is a continuous function in both variables.*

(ii) *For each fixed value of t, $f(\cdot, t)$ is a holomorphic function of the first variable.*

(iii) *The integral*

$$F(z) = \int_a^b f(z, t)\,\mathrm{d}t, \quad z \in \Omega$$

converges uniformly at both limits in any compact set in Ω.

Then F is holomorphic in Ω, and its derivatives of all orders may be found by differentiating under the integral sign.

1.2 Fourier Series

This section is based on [9]. We know that an analytic function f can be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!} \tag{1.2.1}$$

for all values of x within the radius of convergence of the series. In this section we shall be interested in functions which may not be smooth, so that f may not be written in the form (1.2.1). To obtain representations of non smooth functions we need expansions in terms of trigonometric functions.

Definition 1.2.1. *A trigonometric series is one of the form*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.2.2}$$

where a_n, b_n are constants.

Let $f : [-\pi, \pi] \to \mathbb{R}$ be a function. The coefficients a_n, b_n are to be determined in such a way that f is represented by (1.2.2). To do so, we use the so called *orthogonality relations* of the trigonometric functions

$$\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x = \int_{-\pi}^{\pi} \sin mx \sin nx \,\mathrm{d}x = \begin{cases} \pi, & \text{if } m = n, \\ 0, & \text{otherwise}, \end{cases} \tag{1.2.3}$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx \,\mathrm{d}x = 0 \quad \text{for} \quad m, n = 1, 2, \ldots. \tag{1.2.4}$$

Theorem 1.2.1. *Let $f \in C[-\pi, \pi]$. Suppose that*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges uniformly to f for all $x \in [-\pi, \pi]$. Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt \,\mathrm{d}t \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,\mathrm{d}t. \tag{1.2.5}$$

Proof. Let

$$s_k(x) = \frac{1}{2}a_0 + \sum_{m=1}^{k}(a_m \cos mx + b_m \sin mx).$$

Since s_k converges uniformly to f as $k \to \infty$, then for each fixed $n \in \mathbb{N}$, $s_k(x)\cos nx$ converges uniformly to $f(x)\cos nx$ as $k \to \infty$. Note that

$$|s_k(x)\cos nx - f(x)\cos nx| \leqslant |s_k(x) - f(x)|.$$

Therefore, for each fixed $n \in \mathbb{N}$,

$$f(x)\cos nx = \frac{1}{2}a_0 \cos nx + \sum_{m=1}^{\infty}(a_m \cos mx \cos nx + b_m \sin mx \cos nx).$$

Since the series converges uniformly we use Theorem 1.1.1 to integrate term by term between $-\pi$ and π. Then, using (1.2.3) and (1.2.4) we get

$$\int_{-\pi}^{\pi} f(t)\cos nt\,\mathrm{d}t = \pi a_n.$$

Similarly, by repeating the argument for $f(x)\sin(nx)$ we obtain

$$\int_{-\pi}^{\pi} f(t)\sin nt\,\mathrm{d}t = \pi b_n.$$

□

The numbers a_n and b_n are called the *Fourier coefficients* of f. The series (1.2.2) is called the *Fourier series* of f. Let f be any integrable function defined on $[-\pi, \pi]$, then we can always define a_n and b_n by (1.2.5). However, this doesn't guarantee that the Fourier series of f converges pointwise to f. We shall see later a condition that implies the convergence of the Fourier series.

Definition 1.2.2. *Let $f : [a, b] \to \mathbb{R}$ be a function. We say that f is piecewise continuous on $[a, b]$ if and only if*

(i) There exists a partition $\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that f is continuous on each subinterval (x_{k-1}, x_k).

(ii) At each subdivision points $x_0, x_1, \ldots, x_n$ both one-sided limits of f exist.

We denote the set of piecewise continuous functions on $[a, b]$ by $C_{\mathrm{PW}}[a, b]$.

Thus a function $f \in C_{\text{PW}}[a,b]$ has a finite number of discontinuities at $x_0, x_1, \ldots, x_n$. In each of this points the lateral limits

$$\lim_{x \to x_k^-} f(x) \quad \text{and} \quad \lim_{x \to x_k^+} f(x)$$

exist and we denote them by $f(x_k^-)$ and $f(x_k^+)$ respectively. The quantity $f(x_k^+) - f(x_k^-)$ is called the *jump of f at x_k*. We say that f is *standardized* if its values at points of discontinuity are given by

$$f(x_i) = \frac{1}{2}[f(x_k^+) + f(x_k^-)].$$

The *periodic extension* $\widetilde{f}$ of $f \in C_{\text{PW}}[a,b]$ is defined as

$$\widetilde{f}(x) = f(x), \quad a \leqslant x < b,$$

and

$$\widetilde{f}(x + (b-a)) = \widetilde{f}(x), \quad x \in \mathbb{R}$$

then, we standardize $\widetilde{f}$ at a, b and all other points of discontinuity so that $\widetilde{f}$ is defined for all $x \in \mathbb{R}$.

Suppose that we want a Fourier series for f in $J = [0, \pi]$. Since the Fourier coefficients a_n, b_n are given in terms of integrals from $-\pi$ to π, we must somehow change the domain of f to $I = [-\pi, \pi]$. We can do this by defining f arbitrarily on $[-\pi, 0]$; for example $f(x) = \dfrac{1}{\Gamma(x)}$ for $x \in [-\pi, 0]$. Since we are interested in f only on J, properties of convergence of the series on $[-\pi, 0]$ are irrelevant. However, one choice which is useful for many purposes consists in defining f as an *even* or *odd* function on I, since $b_n = 0$ for even functions and $a_n = 0$ for odd functions. Therefore if we define f as an even function, the Fourier series will have cosine terms only, and if we define f as an odd function, the Fourier series will have sine terms only. We call such series a *cosine series* and *sine series* respectively.

Now suppose we want a Fourier series for $f \in C_{\text{PW}}[-L, L]$. For doing this, we introduce a change of variable

$$y = \frac{\pi x}{L}$$

and define $g(y) = f(x)$. Since this transformation maps $[-L, L]$ to $[-\pi, \pi]$, $g \in C_{\text{PW}}[-\pi, \pi]$. Therefore

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny)$$

is a Fourier series of g with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(y) \cos ny \,\mathrm{d}y \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(y) \sin ny \,\mathrm{d}y.$$

Returning to the variable x and the function f we get the formulas for the coefficients a_n, b_n of the modified series:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \,\mathrm{d}x.$$

Therefore

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$$

is a Fourier series for f on the interval $[-L, L]$.

Example 1.2.1. *Let* $f(x) = x - \frac{1}{2}$ *be defined on the interval* $[0,1]$. *We extend* f *as an odd function, as shown in Figure 1.1. Then* f *is standardized so that* $\widetilde{f}(-1) = \widetilde{f}(0) = \widetilde{f}(1) = 0$. *Since* $\widetilde{f}$ *is odd, we have* $a_n = 0$. *Also,*

$$b_n = 2\int_0^1 \left(x - \frac{1}{2}\right) \sin n\pi x \, \mathrm{d}x = \frac{1 + \cos \pi n}{\pi n}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1 + \cos \pi n}{\pi n} \sin \pi n x = \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{\pi n}$$

is a Fourier series for $x - \frac{1}{2}$. *We shall see later that this series converges to* f *on the interval* $[0,1]$.

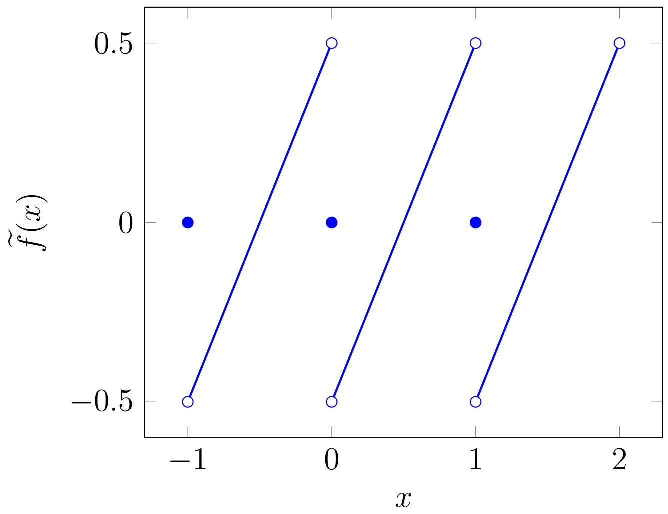

Figure 1.1: Standardized periodic extension of f as an odd function.

It is important to establish a simple criteria determining when a Fourier series converges pointwise, we will show how to obtain large classes of functions with the property that for each value of x in the domain of a function f, the Fourier series converges to $f(x)$. We begin with the following theorem.

Theorem 1.2.2 (Bessel's inequality)**.** *Suppose that* f *is integrable on* $[-\pi, \pi]$. *Let*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

be the Fourier series of f. *Then*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x. \tag{1.2.6}$$

Proof. Let

$$s_n(t) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt).$$

Now write

$$\int_{-\pi}^{\pi} (f(t) - s_n(t))^2 \,\mathrm{d}t = \int_{-\pi}^{\pi} f^2(t) \,\mathrm{d}t - 2\int_{-\pi}^{\pi} f(t)s_n(t) \,\mathrm{d}t + \int_{-\pi}^{\pi} s_n^2(t) \,\mathrm{d}t.$$

From the definition of the Fourier coefficients, we have

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)s_n(t) \,\mathrm{d}t.$$

Also, by multiplying out the terms of $s_n^2(t)$ and using (1.2.3) and (1.2.4) we have

$$\begin{aligned}
\int_{-\pi}^{\pi} s_n^2(t) \,\mathrm{d}t &= \int_{-\pi}^{\pi} \frac{1}{4}a_0^2 \,\mathrm{d}t + \int_{-\pi}^{\pi} \sum_{k=1}^{n}(a_k^2 \cos^2 kt + b_k^2 \sin^2 kt) \,\mathrm{d}t \\
&= \frac{1}{2}a_0^2\pi + \sum_{k=1}^{n}(a_k^2\pi + b_k^2\pi) \\
&= \int_{-\pi}^{\pi} f(t)s_n(t) \,\mathrm{d}t.
\end{aligned}$$

Therefore

$$0 \leqslant \int_{-\pi}^{\pi} (f(t) - s_n(t))^2 \,\mathrm{d}t = \int_{-\pi}^{\pi} f^2(t) \,\mathrm{d}t - \pi\left[\frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right].$$

Since f^2 is integrable we may let $n \to \infty$ and obtain (1.2.6). $\square$

Bessel's inequality shows that a_n and b_n tend to zero as $n \to \infty$ for any function that is square integrable on $[-\pi, \pi]$.

For each $n \in \mathbb{N}$, we define the *Dirichlet kernel* D_n as

$$\mathrm{D}_n(x) = \frac{\sin{(n + \frac{1}{2})x}}{2\sin\frac{x}{2}}.$$

Using the trigonometric identity

$$2\sin\frac{x}{2}\cos kx = \sin{(k + \tfrac{1}{2})x} + \sin{(k - \tfrac{1}{2})x}$$

it follows that

$$\frac{1}{2} + \sum_{k=1}^{n}\cos kx = \frac{\sin{(n + \frac{1}{2})x}}{2\sin\frac{x}{2}} = \mathrm{D}_n(x).$$

Thus the Dirichlet kernel has the following properties

(i) D_n is an even function.

(ii) $\displaystyle\int_{-\pi}^{\pi} \mathrm{D}_n(x) \,\mathrm{d}x = \pi$.

(iii) D_n is a periodic function with period 2π.

Lemma 1.2.1. *Let $f \in C_{\mathrm{PW}}[-\pi,\pi]$ be a periodic function with period 2π, and let*

$$s_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

Then

$$\begin{aligned} s_n(x) - \frac{1}{2}[f(x^+) + f(x^-)] =& \frac{1}{\pi}\int_0^\pi [f(x+u) - f(x^+)]\,\mathrm{D}_n(u)\,\mathrm{d}u \\ &+ \frac{1}{\pi}\int_0^\pi [f(x-u) - f(x^-)]\,\mathrm{D}_n(u)\,\mathrm{d}u. \end{aligned}$$

Proof. We have

$$\begin{aligned} s_n(x) &= \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}(\cos kt \cos kx + \sin kt \sin kx)\right]\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right]\mathrm{d}t. \end{aligned}$$

The change of variable $u = t - x$ gives

$$s_n(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)\,\mathrm{D}_n(u)\,\mathrm{d}u.$$

Since f and D_n are periodic functions with period 2π, we may change the interval of integration to $-\pi \leqslant u \leqslant \pi$. Then

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{0} f(x+v)\,\mathrm{D}_n(v)\,\mathrm{d}v + \frac{1}{\pi}\int_0^\pi f(x+u)\,\mathrm{D}_n(u)\,\mathrm{d}v.$$

In the first integral we replace v by $-u$ and recall that D_n is even to obtain

$$s_n(x) = \frac{1}{\pi}\int_0^\pi [f(x+u) - f(x-u)]\,\mathrm{D}_n(u)\,\mathrm{d}u.$$

Now we use the property

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{D}_n(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^\pi \mathrm{D}_n(x)\,\mathrm{d}x = 1$$

to get

$$\begin{aligned} s_n(x) - \frac{1}{2}[f(x^+) + f(x^-)] =& \frac{1}{\pi}\int_0^\pi [f(x+u) - f(x^+)]\,\mathrm{D}_n(u)\,\mathrm{d}u \\ &+ \frac{1}{\pi}\int_0^\pi [f(x-u) - f(x^-)]\,\mathrm{D}_n(u)\,\mathrm{d}u. \end{aligned}$$

□

Now we are ready to give a simple criteria determining when a Fourier series converges, first we need the following definition.

Definition 1.2.3. *Let $f : [a, b] \to \mathbb{R}$ be a function. We say that f is piecewise smooth on $[a, b]$ if and only if*

(i) $f \in C_{\mathrm{PW}}[-\pi, \pi]$ *and*

(ii) With the notation of Definition 1.2.2, f' exists in (x_{k-1}, x_k) and $f' \in C(x_{k-1}, x_k)$.

The following theorem gives a sufficient condition for the convergence of a Fourier series.

Theorem 1.2.3. *Let f be a piecewise smooth function on $[-\pi, \pi]$, standardized, and periodic with period 2π. Then the Fourier series of f converges point-wise for all $x \in [-\pi, \pi]$.*

Proof. Since f is standardized, we write Lemma 1.2.1 in the form

$$s_n(x) - f(x) = \mathscr{I}_n(x) + \mathscr{J}_n(x),$$

where

$$\mathscr{I}_n(x) = \frac{1}{\pi}\int_0^\pi [f(x+u) - f(x^+)]\,\mathrm{D}_n(u)\,\mathrm{d}u \quad \text{and} \quad \mathscr{J}_n(x) = \frac{1}{\pi}\int_0^\pi [f(x-u) - f(x^-)]\,\mathrm{D}_n(u)\,\mathrm{d}u.$$

Then

$$\begin{aligned}\mathscr{J}_n(x) &= \frac{1}{\pi}\int_0^\pi \frac{f(x-u) - f(x^-)}{2\sin\frac{u}{2}} \sin{(n+\tfrac{1}{2})u}\,\mathrm{d}u \\ &= \frac{1}{\pi}\int_0^\pi \frac{f(x-u) - f(x^-)}{2\sin\frac{u}{2}} (\sin nu \cos\frac{u}{2} + \cos nu \sin\frac{u}{2})\,\mathrm{d}u.\end{aligned}$$

Let

$$g_1(x,u) = \frac{f(x-u) - f(x^-)}{2\sin\frac{u}{2}}\cos\frac{u}{2} \quad \text{and} \quad g_2(x,u) = \frac{1}{2}[f(x-u) - f(x^-)].$$

By L'Hôpital's rule,

$$g_1(x, 0^+) = -f'(x^-).$$

Since f is piecewise smooth, g_1 and g_2 are piecewise continuous functions, they are integrable. Now write $\mathscr{J}_n$ in the form

$$\mathscr{J}_n(x) = \frac{1}{\pi}\int_0^\pi (g_1(x,u)\sin nu + g_2(x,u)\cos nu)\,\mathrm{d}u = \hat{a}_n + \hat{b}_n.$$

where $\hat{a}_n$ and $\hat{b}_n$ are the nth Fourier coefficients of $\frac{1}{2}g_1$ and $\frac{1}{2}g_2$ respectively. By Bessel's inequality, $\hat{a}_n, \hat{b}_n \to 0$ as $n \to \infty$. Thus $\mathscr{J}_n(x) \to 0$ as $n \to \infty$. Similarly $\mathscr{I}_n(x) \to 0$ as $n \to \infty$. Therefore

$$s_n(x) - f(x) \xrightarrow[n\to\infty]{} 0.$$

□

As a corollary, the Fourier series of the function f in Example 1.2.1 converges to f, so

$$x - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{\pi n}, \qquad 0 \leqslant x \leqslant 1.$$

So far we've been working with the Fourier series of real-valued functions, but we can extend the results of this section to complex-valued functions as follows: Let $f\colon [-\pi,\pi] \to \mathbb{C}$ be a function. Then $f(t) = u(t) + iv(t)$, where u, v are real-valued functions. Let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{and} \quad \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx)$$

be the Fourier series of u and v respectively. Naturally we define the Fourier series of f to be

$$\frac{1}{2}(a_0 + i\alpha_0) + \sum_{n=1}^{\infty}[(a_n + i\alpha_n)\cos nx + (b_n + i\beta_n)\sin nx].$$

For each $n \in \mathbb{Z}$ define

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-int}\,\mathrm{d}t.$$

Using the definition of a_n, b_n, α_n and β_n, it is easy to see that

$$c_n = \frac{1}{2}(a_n + i\alpha_n) + \frac{1}{2}(\beta_n - ib_n).$$

Therefore

$$\begin{aligned}\sum_{n\in\mathbb{Z}} c_n\,\mathrm{e}^{inx} =& \frac{1}{2}(a_0 + i\alpha_0) + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{2}a_n\cos nx + \frac{1}{2}i\alpha_n\cos nx + \frac{1}{2}\beta_n\cos nx - \frac{1}{2}ib_n\cos nx\\ &+ \frac{1}{2}ia_n\sin nx - \frac{1}{2}\alpha_n\sin nx + \frac{1}{2}i\beta_n\sin nx + \frac{1}{2}b_n\sin nx.\end{aligned}$$

Using the fact that if g is an even function and h is an odd function then

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} g(n) = 2\sum_{n=1}^{\infty} g(n) \quad \text{and} \quad \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} h(n) = 0,$$

we obtain

$$\sum_{n\in\mathbb{Z}} c_n\,\mathrm{e}^{inx} = \frac{1}{2}(a_0 + i\alpha_0) + \sum_{n=1}^{\infty}[(a_n + i\alpha_n)\cos nx + (b_n + i\beta_n)\sin nx].$$

This shows that the Fourier series of f is actually

$$\sum_{n\in\mathbb{Z}} c_n\,\mathrm{e}^{inx}.$$

1.3 Bernoulli Numbers

It is know that

$$\begin{aligned}\sum_{i=1}^{n-1} i &= \frac{1}{2}n^2 - \frac{1}{2}n,\\ \sum_{i=1}^{n-1} i^2 &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n,\\ \sum_{i=1}^{n-1} i^3 &= \frac{1}{4}n^4 - \frac{1}{3}n^3 + \frac{1}{4}n^2,\\ \sum_{i=1}^{n-1} i^4 &= \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n,\end{aligned}$$

and so on. Bernoulli had a particular interest in the coefficients of n, that is $-\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, \ldots$. Euler called this numbers the *Bernoulli numbers* $\hat{B}_1, \hat{B}_2, \hat{B}_3, \hat{B}_4, \ldots$. The modern definition of the Bernoulli numbers uses a *generating function*.

Definition 1.3.1. *Let F be an analytic function in a neighborhood of 0. We say that F is a generating function for the sequence of real numbers $\{a_n\}_{n\geqslant 0}$ if*

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

for all x within the radius of convergence of the series.

For example, a generating function for the constant sequence $1, 1, 1, \ldots$ is $F(x) = \frac{1}{1-x}$, since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Definition 1.3.2. *The Bernoulli numbers are the sequence of real numbers $\{B_n\}_{n\in\mathbb{N}}$ defined by the generating function*

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi.$$

We will show (Theorem 1.3.1) that the modern definition agrees with Euler's definition, that is

$$\hat{B}_n = B_n, \quad n \geqslant 1.$$

Lemma 1.3.1. *The function f defined by*

$$f(z) = \frac{z}{\mathrm{e}^z - 1} - 1 + \frac{1}{2}z$$

is even.

Proof. We have

$$f(z) = \frac{z}{2}\coth\frac{z}{2} - 1,$$

since coth is an odd function the result follows. □

By expanding the function $\frac{z}{\mathrm{e}^z - 1}$ as a power series, we obtain the first Bernoulli numbers:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}.$$

Therefore, by Lemma 1.3.1 all Bernoulli numbers with odd index greater than 2 vanishes, that is

$$B_{2n+1} = 0, \quad n \geqslant 1.$$

Definition 1.3.3. *The Bernoulli polynomials are defined by the generating function*

$$\frac{z\,\mathrm{e}^{xz}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \mathrm{B}_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi. \tag{1.3.1}$$

The first Bernoulli polynomials are

$$\begin{aligned}\mathrm{B}_0(x) &= 1,\\ \mathrm{B}_1(x) &= x - \frac{1}{2},\\ \mathrm{B}_2(x) &= x^2 - x + \frac{1}{6}.\end{aligned}$$

By definition of the Bernoulli polynomials, note that

$$\mathrm{B}_n(0) = B_n. \tag{1.3.2}$$

Now we will show that the Bernoulli numbers appear in the sum $\sum_{i=0}^{n-1} i^p$, for this purpose we need some properties of the Bernoulli numbers and the Bernoulli polynomials.

We have

$$\int_0^1 \frac{z\,\mathrm{e}^{xz}}{\mathrm{e}^z - 1}\,\mathrm{d}x = \frac{\mathrm{e}^z}{\mathrm{e}^z - 1} - \frac{1}{\mathrm{e}^z - 1} = 1.$$

Now we use (1.3.1) and Theorem 1.1.1 to interchange summation and integration. Thus

$$\sum_{n=0}^{\infty} \left(\int_0^1 \mathrm{B}_n(x)\,\mathrm{d}x \right) \frac{z^n}{n!} = 1,$$

and so

$$\int_0^1 \mathrm{B}_n(x)\,\mathrm{d}x = 0, \quad n \geqslant 1. \tag{1.3.3}$$

By differentiating (1.3.1) with respect to x, we get

$$\frac{z^2\,\mathrm{e}^{xz}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \mathrm{B}_n'(x) \frac{z^n}{n!},$$

now we multiply by z (1.3.1) and obtain

$$\frac{z^2\,\mathrm{e}^{xz}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \mathrm{B}_n(x) \frac{z^{n+1}}{n!}.$$

Putting together the above equations, we get

$$\mathrm{B}_n'(x) = n\,\mathrm{B}_{n-1}(x). \tag{1.3.4}$$

Another important identity is obtained as follows: Using (1.3.1), we have

$$z\,\mathrm{e}^{xz} = \sum_{n=0}^{\infty} [\mathrm{B}_n(x+1) - \mathrm{B}_n(x)] \frac{z^n}{n!}.$$

Since

$$z\,\mathrm{e}^{xz} = \sum_{n=0}^{\infty} x^n \frac{z^{n+1}}{n!},$$

we obtain

$$\mathrm{B}_n(x+1) - \mathrm{B}_n(x) = nx^{n-1}. \tag{1.3.5}$$

For the next lemma, we recall how to multiply two series:

$$\left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{n=0}^{\infty} b_n z^n \right) = \sum_{n=0}^{\infty} c_n z^n \quad \text{where} \quad c_n = \sum_{j=0}^{n} a_j b_{n-j}. \tag{1.3.6}$$

Lemma 1.3.2. *Let $x, y \in \mathbb{R}$, then*

$$\mathrm{B}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{B}_k(x) y^{n-k}.$$

Proof. We have

$$\sum_{n=0}^{\infty} \mathrm{B}_n(x+y) \frac{z^n}{n!} = \left(\frac{z\, \mathrm{e}^{xz}}{\mathrm{e}^z - 1} \right) \mathrm{e}^{yz} = \left(\sum_{n=0}^{\infty} \frac{\mathrm{B}_n(x)}{n!} z^n \right) \left(\sum_{n=0}^{\infty} \frac{y^n}{n!} z^n \right).$$

Now we use (1.3.6) to get

$$\sum_{n=0}^{\infty} \mathrm{B}_n(x+y) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \mathrm{B}_k(x) y^{n-k} \right) \frac{z^n}{n!},$$

and the result follows. □

Note the similarity between Lemma 1.3.2 and the Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Since $\mathrm{B}_n(0) = B_n$, by Lemma 1.3.2 it follows that

$$\mathrm{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{1.3.7}$$

Now, using (1.3.4) we obtain the identity

$$\int_x^y \mathrm{B}_p(t)\, \mathrm{d}t = \frac{1}{p+1} \int_x^y \mathrm{B}'_{p+1}(t)\, \mathrm{d}t = \frac{1}{p+1} [\mathrm{B}_{p+1}(y) - \mathrm{B}_{p+1}(x)]. \tag{1.3.8}$$

In the particular case $y = x + 1$, we use (1.3.5) to obtain

$$\int_x^{x+1} \mathrm{B}_p(t)\, \mathrm{d}t = x^p. \tag{1.3.9}$$

Now we are ready to show the connection between Bernoulli numbers and the numbers that appear in the sum $\sum_{i=0}^{n-1} i^p$.

Theorem 1.3.1 (Faulhaber's formula)**.** *Let $p \in \mathbb{N}$, then*

$$\sum_{i=0}^{n-1} i^p = \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} B_k n^{p+1-k}.$$

Proof. Let $p \in \mathbb{N}$. We use (1.3.9) and (1.3.8) to get

$$\sum_{i=0}^{n-1} i^p = \sum_{i=0}^{n-1} \int_i^{i+1} \mathrm{B}_p(t)\, \mathrm{d}t = \int_0^n \mathrm{B}_p(t)\, \mathrm{d}t = \frac{1}{p+1} [\mathrm{B}_{p+1}(n) - \mathrm{B}_{p+1}(0)].$$

Now we use the identity $\mathrm{B}_{p+1}(0) = B_{p+1}$ and (1.3.7) to obtain

$$\sum_{i=0}^{n-1} i^p = \frac{1}{p+1} \left[\sum_{k=0}^{p+1} \binom{p+1}{k} B_k x^{p+1-k} - B_{p+1} \right] = \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} B_k n^{p+1-k}.$$ □

Property (1.3.8) can be used to prove the following theorem

Theorem 1.3.2. *Let $n \in \mathbb{N}$, then for every $x \in [0,1]$ we have*

$$\mathrm{B}_{2n-1}(x) = 2(-1)^{n+1}(2n-1)! \sum_{m=1}^{\infty} \frac{\sin(2\pi mx)}{(2\pi m)^{2n-1}}. \tag{1.3.10}$$

Proof. The proof runs by induction on n. For $n = 1$, we use Example 1.2.1 to obtain

$$\mathrm{B}_1(x) = x - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{\pi n}, \qquad 0 \leqslant x \leqslant 1.$$

Now suppose (1.3.10) holds for $n \in \mathbb{N}$. Using (1.3.4) twice, we have

$$\mathrm{B}''_{2n+1}(x) = (2n+1)2n\,\mathrm{B}_{2n-1}(x) = 2(-1)^{n+1}(2n+1)! \sum_{m=1}^{\infty} \frac{\sin(2\pi mx)}{(2\pi m)^{2n-1}}.$$

Integrating twice the above equation and using Theorem 1.1.1 to interchange summation and integration we get

$$\mathrm{B}_{2n+1}(x) = 2(-1)^{n+2}(2n+1)! \sum_{m=1}^{\infty} \frac{\sin(2\pi mx)}{(2\pi m)^{2n+1}}.$$ □

Corollary 1.3.2.1. *Let $n \in \mathbb{N}$, then for every $x \in [0,1]$ we have*

$$\mathrm{B}_{2n}(x) = 2(-1)^{n+1}(2n)! \sum_{m=1}^{\infty} \frac{\cos(2\pi mx)}{(2\pi m)^{2n}}. \tag{1.3.11}$$

Proof. A single integration of (1.3.10) yields (1.3.11). □

The special case $x = 0$ in (1.3.11) gives an interesting result for the Bernoulli numbers:

$$B_{2n} = 2(-1)^{n+1}(2n)! \sum_{m=1}^{\infty} \frac{1}{(2\pi m)^{2n}}. \tag{1.3.12}$$

It is of interest, since with this result we can express the Riemann zeta function (which will be discussed in Chapter 3) of positive even integers in terms of Bernoulli numbers, that is

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!},$$

for $n = 1, 2, 3, \ldots$.

1.4 Introduction to Asymptotic Analysis

1.4.1 Origin of Asymptotic Expansions

This section is based on [10]. Consider the integral

$$F(x) = \int_0^{\infty} \mathrm{e}^{-xt} \cos t \,\mathrm{d}t \quad (x > 1).$$

We can try to evaluate this integral by expanding $\cos t$ using it's Taylor series around $t = 0$ and then integrating the resulting series term by term. We obtain

$$F(x) = \int_0^\infty \mathrm{e}^{-xt} \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots \right) \mathrm{d}t = \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} - \cdots .$$

Since $x > 1$ the last series converges to

$$F(x) = \frac{x}{x^2+1}.$$

This result is correct and can be confirmed by means of two integration by parts. Now let us follow the same procedure with the integral

$$G(x) = \int_0^\infty \frac{\mathrm{e}^{-xt}}{1+t} \mathrm{d}t.$$

We obtain

$$G(x) = \int_0^\infty \mathrm{e}^{-xt}(1 - t + t^2 - \cdots) \, \mathrm{d}t = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots . \tag{1.4.1}$$

This series diverges for all $x \in \mathbb{R}$, and therefore appears to be meaningless. One might ask why did the procedure succeed in the first case but not in the second, and the answer is given by Theorem 1.1.1. In the first case, the expansion of $\cos t$ converges uniformly in any compact interval in $(0, \infty)$, whereas in the second case the expansion of $(1 + t)^{-1}$ diverges when $t \geqslant 1$.

However, not everything is lost. Suppose we try to sum the series (1.4.1) numerically for a particular value of x, say $x = 15$. The first four terms are given by

$$\frac{1}{15} - \frac{1!}{15^2} + \frac{2!}{15^3} - \frac{3!}{15^4} = 0.0626963.$$

Surprisingly enough, this is very close to the correct value $G(15) = 0.0627203\ldots$. To investigate this unexpected result, we consider the difference $R_m(x)$ between $G(x)$ and the mth partial sum of (1.4.1):

$$\begin{aligned} R_m(x) &= \int_0^\infty \frac{\mathrm{e}^{-xt}}{1+t} \mathrm{d}t - \sum_{n=1}^{m} (-1)^{n-1} \frac{(n-1)!}{x^n} \\ &= (-1)^m \int_0^\infty \frac{t^m \, \mathrm{e}^{-xt}}{1+t} \mathrm{d}t, \end{aligned}$$

where the last equality follows from the fact that for all $m \in \mathbb{N}$,

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots (-1)^{m-1} t^{m-1} + (-1)^m \frac{t^m}{1+t}.$$

Therefore,

$$|R_m(x)| < \int_0^\infty t^m \, \mathrm{e}^{-xt} \, \mathrm{d}t = \frac{m!}{x^{m+1}}.$$

This means that the partial sums of (1.4.1) approximate $G(x)$ with an error that is smaller than the *first neglected term* of the series. The series (1.4.1) is an example of an *asymptotic expansion*, the definition is due to Poincaré and will be discussed in the next section.

1.4.2 Asymptotic Notation

In order to describe the behaviour of a function f at infinity in terms of a known function g we shall use the following useful notation, due to Bachmann and Landau.

Definition 1.4.1. *Let f, g be real-valued functions and let $a \geqslant 0$ or $a = \pm\infty$.*

- *If* $\lim_{x\to a} \dfrac{f(x)}{g(x)} = 1$, *we write* $f(x) \sim g(x) \quad (x \to a)$.
- *If* $\lim_{x\to a} \dfrac{f(x)}{g(x)} = 0$, *we write* $f(x) = o(g(x)) \quad (x \to a)$.
- *If there exists $C > 0$ such that $|f(x)| \leqslant C|g(x)|$ for all x in a neighborhood of a, we write* $f(x) = \mathcal{O}(g(x)) \quad (x \to a)$.

So, for example, if we want to say that f vanishes as $x \to a$ we shall write $f(x) = o(1) \quad (x \to a)$. If we write $f(x) = \mathcal{O}(1) \quad (x \to a)$ this simply means that f is bounded in a neighborhood of a. As simple examples,

$$(x^3 + 2x^2 - x)^2 \sim x^6 \quad (x \to \infty), \qquad \sin x \sim x \quad (x \to 0).$$

We can extend Definition 1.4.1 to complex-valued functions as follows: Let S be any set, let f and φ be complex-valued functions defined on S. Suppose there exists $K > 0$ such that

$$|f(s)| \leqslant K|\varphi(s)| \qquad \forall s \in S.$$

Then, we say that

$$f(s) = \mathcal{O}(\varphi(s)) \qquad (s \in S).$$

Our first example is provided by the tail of a convergent power series:

Theorem 1.4.1. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a convergent series with radius of convergence R. Then for fixed m,*

$$\sum_{n=m}^{\infty} a_n z^n = \mathcal{O}(z^m)$$

in any disk $|z| \leqslant r$ such that $r < R$.

Proof. Let $\delta > 0$ such that $r < \delta < R$. Since $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$, there exists $C > 0$ such that

$$|a_n|\delta^n \leqslant C \quad (n \in \mathbb{N}).$$

Therefore,

$$\left|\sum_{n=m}^{\infty} a_n z^n\right| \leqslant \sum_{n=m}^{\infty} C\frac{|z|^n}{\delta^n} = \frac{C\delta^{1-m}|z|^m}{\delta - |z|} \leqslant \frac{C\delta^{1-m}}{\delta - R}|z|^m,$$

and the result follows. □

Sometimes, it is useful to write $f \ll g$ instead of $f(x) = \mathcal{O}(g(x))$[1]. We shall use this notation throughout this document.

[1] This is Vinogradov's notation and it will prove to be quite useful.

Example 1.4.1. *Let $\varepsilon > 0$. Then, by L'Hôpital's rule we have*

$$\lim_{x\to\infty} \frac{\log x}{x^\varepsilon} = \lim_{x\to\infty} \frac{1}{\varepsilon x^\varepsilon} = 0.$$

This shows that

$$\log x \ll x^\varepsilon \qquad (x \to \infty)$$

for every $\varepsilon > 0$.

1.4.3 Asymptotic Expansions

Now we are ready to give the definition of an asymptotic expansion and mention a very useful result known as *Watson's Lemma.*

Definition 1.4.2 (Poincaré (1886))**.** *Let F be a function of a real or complex variable z; let $\sum_{n=0}^{\infty} a_n z^{-n}$ denote a (convergent or divergent) formal power series, of which the sum of the first n terms is denoted by $S_n(z)$; let*

$$R_n(z) = F(z) - S_n(z).$$

That is,

$$F(z) = a_0 + \frac{a_1}{z} + \cdots + \frac{a_{n-1}}{z^{n-1}} + R_n(z), \quad (n \in \mathbb{N}),$$

where we assume that when $n = 0$ we have $F(z) = R_0(z)$. Now, suppose that for each $n \in \mathbb{N}$ the following relation holds

$$R_n(z) = \mathcal{O}(z^{-n}) \quad (z \to \infty)$$

in some unbounded region Δ. Then $\sum_{n=0}^{\infty} a_n z^{-n}$ is called an asymptotic expansion of the function F and we denote this by [2]

$$F(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \quad (z \to \infty), \quad z \in \Delta.$$

Example 1.4.2. *The classical example is the exponential integral $\mathrm{E}_1(x)$ (which will be discussed in Chapter 3):*

$$F(x) = x \int_x^\infty \frac{\mathrm{e}^{x-t}}{t}\,\mathrm{d}t = x\,\mathrm{e}^x\,\mathrm{E}_1(x) \quad (x > 0).$$

Repeated integration by parts with $g(t) = \dfrac{1}{t}$ and $h(t) = \mathrm{e}^{x-t}$ as in Theorem 1.1.2 yields

$$F(x) = \sum_{k=0}^{n-1} \frac{(-1)^k k!}{x^k} + (-1)^n n! x \int_x^\infty \frac{\mathrm{e}^{x-t}}{t^{n+1}}\,\mathrm{d}t.$$

Therefore,

$$|R_n(x)| = n! x \int_x^\infty \frac{\mathrm{e}^{x-t}}{t^{n+1}}\,\mathrm{d}t \leqslant \frac{n!}{x^n} \int_x^\infty \mathrm{e}^{x-t}\,\mathrm{d}t = \frac{n!}{x^n}.$$

This means that for each $n \in \mathbb{N}$ we have

$$R_n(x) = \mathcal{O}(x^{-n}) \quad (x \to \infty).$$

Hence

$$F(x) \sim \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^n} \quad (x \to \infty).$$

[2]Note that symbol $\sim$ is now being used in two different ways. It should be clear from the context when $\sim$ refers to an asymptotic expansion and when it refers to an asymptotic approximation

In the above example we can find the asymptotic expansion in a different way. Using the substitution $t = x(1+u)$ we can write $F(x)$ as a Laplace integral [3]:

$$F(x) = x \int_0^\infty \mathrm{e}^{-xu} f(u)\, \mathrm{d}u, \quad f(u) = \frac{1}{1+u}.$$

We have already encountered this integral in (1.4.1). Proceeding as in (1.4.1) we write

$$f(u) = 1 - u + u^2 - \cdots (-1)^{n-1} u^{n-1} + (-1)^n \frac{u^n}{1+u},$$

and we obtain exactly the same expansion, with the same expression and upper bound for $|R_n(x)|$.

Now we introduce the *Watson's lemma*, which is probably the most frequently used result for deriving asymptotic expansions. For a proof we refer to [10].

Theorem 1.4.2 (Watson's Lemma)**.** *Suppose that*

(i) f is a real or complex-valued function of the positive real variable t with a finite number of discontinuities and infinities.

(ii) As $t \to 0^+$, there exists $\lambda \in \mathbb{C}$ such that

$$f(t) \sim t^{\lambda-1} \sum_{n=0}^{\infty} a_n t^n, \quad \mathfrak{Re}(\lambda) > 0.$$

(iii) The integral

$$F(z) = \int_0^\infty \mathrm{e}^{-zt} f(t)\, \mathrm{d}t$$

is convergent for sufficiently large values of $\mathfrak{Re}(z)$.

Then

$$F(z) \sim \sum_{n=0}^{\infty} \Gamma(n+\lambda) \frac{a_n}{z^{n+\lambda}} \quad (z \to \infty),$$

in the sector $|\arg z| \leqslant \frac{1}{2}\pi - \delta$, for some $0 < \delta < \frac{\pi}{2}$ and $z^{n+\lambda}$ takes its principal value. Here

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t}\, t^{z-1}\, \mathrm{d}t \qquad (\mathfrak{Re}(z) > 0),$$

denotes the Gamma function which will be discussed in Chapter 2.

A larger sector for $\arg z$ can be obtained when we know that f is analytic in a certain domain of the complex plane. For example, when f is analytic in the sector $|\arg z| < \frac{\pi}{2}$ and $f(t) = \mathcal{O}(\mathrm{e}^{\sigma t})$ in that sector, for some number σ, then the asymptotic expansion in Watson's lemma holds in the sector $|\arg z| \leqslant \pi - \delta$ for some $0 < \delta < \pi$.

[3] A Laplace integral is one of the form $\int_a^b f(t)\, \mathrm{e}^{-st}\, \mathrm{d}t$, where s is a real or complex number.

1.5 The Fourier Transform

We denote by L^1 the set of functions whose absolute value has finite integral, that is

$$L^1(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C} \quad \text{such that} \quad \int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty \right\}.$$

Definition 1.5.1. *Let $f \in L^1(\mathbb{R})$. The Fourier transform of f is the function $\hat{f}\colon \mathbb{R} \to \mathbb{C}$ defined by*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi i \xi t} f(t)\,\mathrm{d}t.$$

Another notation for the Fourier transform of f is $\mathscr{F} f := \hat{f}$.

Let $f \in C^\infty(\mathbb{R}, \mathbb{C})$. We say that f is a *Schwartz function* if for every $m, n \in \mathbb{N}$ we have

$$\sup_{t \in \mathbb{R}} |t^m f^{(n)}(t)| < \infty.$$

Intuitively, we are saying that f and all its derivatives decrease faster than any polynomial. Note that if f is a Schwartz function, then

$$\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t \ll \int_{-\infty}^{\infty} \frac{1}{t^2+1}\,\mathrm{d}t = \pi < \infty.$$

Therefore, we can take the Fourier transform of Schwartz functions.

Theorem 1.5.1 (Poisson summation formula)**.** *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a Schwartz function. Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

Proof. Consider the function $F : \mathbb{R} \to \mathbb{C}$ defined by $F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$. Since F is a smooth periodic function with period 1, its Fourier series expansion converges pointwise to F, so

$$F(x) = \sum_{n \in \mathbb{Z}} c_n \,\mathrm{e}^{2\pi i n x},$$

where

$$c_n = \int_0^1 F(x)\,\mathrm{e}^{-2\pi i n x}\,\mathrm{d}x = \int_0^1 \sum_{m \in \mathbb{Z}} f(x+m)\,\mathrm{e}^{-2\pi i n x}\,\mathrm{d}x.$$

Since f is a Schwartz function, with the aid of Theorem 1.1.1 we can interchange summation and integration to get

$$c_n = \sum_{m \in \mathbb{Z}} \int_0^1 f(x+m)\,\mathrm{e}^{-2\pi i n x}\,\mathrm{d}x.$$

Letting $t = x + m$ we obtain

$$c_n = \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(t)\,\mathrm{e}^{-2\pi i n t}\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-2\pi i n t}\,\mathrm{d}t = \hat{f}(n).$$

Therefore

$$\sum_{n \in \mathbb{Z}} f(x+n) = F(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n)\,\mathrm{e}^{2\pi i n x},$$

and the result follows by evaluating at $x = 0$. $\square$

We shall need the following theorem from complex analysis, for a proof see [6].

Theorem 1.5.2 (Uniqueness principle). *Let D be a domain and let $f, g : D \subseteq \mathbb{C} \to \mathbb{C}$ be analytic functions on D. If $f(z) = g(z)$ for z belonging to a set that has a nonisolated point, then $f(z) = g(z)$ for all $z \in D$.*

For the next Lemma we recall how to evaluate the *Gaussian integral*. Take $z = \frac{1}{2}$ in the Euler's reflection formula (2.1.5) to obtain

$$\Gamma\Big(\frac{1}{2}\Big) = \int_0^\infty \frac{\mathrm{e}^{-t}}{\sqrt{t}}\,\mathrm{d}t = \sqrt{\pi},$$

or equivalently

$$\int_0^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u = \frac{\sqrt{\pi}}{2}. \tag{1.5.1}$$

Lemma 1.5.1. *For every $z, w \in \mathbb{C}$ with $\mathfrak{Re}(z) > 0$, we have*

$$\int_{-\infty}^\infty \mathrm{e}^{-zt^2+2wt}\,\mathrm{d}t = \sqrt{\frac{\pi}{z}}\,\mathrm{e}^{\frac{w^2}{z}},$$

where we take the principal branch of the square root.

Proof. Let $x > 0$, $y \in \mathbb{R}$ and let $u = \sqrt{x}\,t - \frac{y}{\sqrt{x}}$. Then

$$\begin{aligned}\int_{-\infty}^\infty \mathrm{e}^{-xt^2+2yt}\,\mathrm{d}t &= \mathrm{e}^{\frac{y^2}{x}} \int_{-\infty}^\infty \mathrm{e}^{-(\sqrt{x}\,t-\frac{y}{\sqrt{x}})^2}\,\mathrm{d}t \\ &= \frac{\mathrm{e}^{\frac{y^2}{x}}}{\sqrt{x}} \int_{-\infty}^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u. \\ &= \sqrt{\frac{\pi}{x}}\,\mathrm{e}^{\frac{y^2}{x}},\end{aligned}$$

where the last equality follows from (1.5.1) and the symmetry of e^{-u^2}. Now let $z, w \in \mathbb{C}$ with $\mathfrak{Re}(z) > 0$. By the uniqueness principle we obtain

$$\int_{-\infty}^\infty \mathrm{e}^{-zt^2+2wt}\,\mathrm{d}t = \sqrt{\frac{\pi}{z}}\,\mathrm{e}^{\frac{w^2}{z}},$$

where we take the principal branch of the square root. This completes the proof ∎

Corollary 1.5.2.1. *The Schwartz function $g\colon \mathbb{R} \to [1, \infty)$ defined by $g(t) = \mathrm{e}^{-\pi t^2}$ is its own Fourier transform.*

Proof. Taking $z = \pi$ and $w = -i\pi\xi$ in Lemma 1.5.1 we get

$$\hat{g}(\xi) = \int_{-\infty}^\infty \mathrm{e}^{-2\pi i\xi t}\,\mathrm{e}^{-\pi t^2}\,\mathrm{d}t = \mathrm{e}^{-\pi\xi^2}.$$ ∎

Let $\lambda \in \mathbb{R}$. It is straightforward to verify that

$$\widehat{f(\lambda t)}(\xi) = \frac{1}{\lambda}\hat{f}\Big(\frac{\xi}{\lambda}\Big). \tag{1.5.2}$$

Now we are ready to prove the following theorem:

Theorem 1.5.3. *The Jacobi theta function $\theta : (0, \infty) \to \mathbb{R}$ defined by*

$$\theta(x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\pi n^2 x},$$

satisfies the functional equation

$$\theta\Big(\frac{1}{x}\Big) = \sqrt{x}\,\theta(x).$$

Proof. Let $f_t(x) = \mathrm{e}^{-\pi x^2 t}$. Note that $f_t(x) = f_1(\sqrt{t}\,x)$. Then, by (1.5.2) we have

$$\hat{f}_t(\xi) = \widehat{f_1(\sqrt{t}x)}(\xi) = \frac{1}{\sqrt{t}}\hat{f}_1\Big(\frac{\xi}{\sqrt{t}}\Big).$$

But Lemma (1.5.2.1) states that $\hat{f}_1 = f_1$. Therefore

$$\hat{f}_t(\xi) = \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{\pi\xi^2}{t}}.$$

Now we apply Poisson summation (Theorem 1.5.1) to get

$$\theta(t) = \sum_{n\in\mathbb{Z}} f_t(n) = \sum_{n\in\mathbb{Z}} \hat{f}_t(n) = \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{\pi n^2}{t}} = \frac{1}{\sqrt{t}}\,\theta\Big(\frac{1}{t}\Big). \qquad \square$$

Now we prove a theorem that essentially says that for many types of functions it is possible to recover them from its Fourier transform.

Theorem 1.5.4 (Fourier inversion theorem)**.** *Let $f, \hat{f} \in L^1(\mathbb{R})$, with f continuous. Then*

$$f(x) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi i\xi x}\,\hat{f}(\xi)\,\mathrm{d}\xi.$$

Proof. Let us show that

$$f(x) = \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi i\xi t}\,f(t)\,\mathrm{d}t\Big)\,\mathrm{e}^{2\pi i\xi x}\,\mathrm{d}\xi.$$

It is very tempting to exchange the order of integration in the above expression, but we can't do this because the function $f(t)\,\mathrm{e}^{2\pi i\xi(x-t)}$ is not $L^1(\mathbb{R}\times\mathbb{R})$. We use the following trick: Given $\varepsilon > 0$, let

$$\mathscr{I}_\varepsilon(x) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{2\pi i\xi(x-t)}\,\mathrm{e}^{-\varepsilon^2\xi^2}\,\mathrm{d}t\,\mathrm{d}\xi.$$

By definition of Fourier transform, we have

$$\mathscr{I}_\varepsilon(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\,\mathrm{e}^{2\pi i\xi x}\,\mathrm{e}^{-\varepsilon^2\xi^2}\,\mathrm{d}\xi.$$

Since

$$|\hat{f}(\xi)\,\mathrm{e}^{2\pi i\xi x}\,\mathrm{e}^{-\varepsilon^2\xi^2}| \leqslant |\hat{f}(\xi)|,$$

and $\hat{f} \in L^1(\mathbb{R})$, then

$$\mathscr{I}_\varepsilon(x) \xrightarrow[\varepsilon\to 0]{} \int_{-\infty}^{\infty} \mathrm{e}^{2\pi i\xi x}\,\hat{f}(\xi)\,\mathrm{d}\xi. \tag{1.5.3}$$

Now, since $f(t)\,\mathrm{e}^{2\pi i\xi(x-t)}\,\mathrm{e}^{-\varepsilon^2\xi^2}$ is $L^1(\mathbb{R}\times\mathbb{R})$ we can apply Fubini's theorem to interchange the order of integration. Thus

$$\mathscr{I}_\varepsilon(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{2\pi i\xi(x-t)}\,\mathrm{e}^{-\varepsilon^2\xi^2}\,f(t)\,\mathrm{d}\xi\,\mathrm{d}t.$$

Now we use Lemma 1.5.1 with $z=\varepsilon^2$, $w=i\pi(x-t)$ to obtain

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi i\xi(x-t)}\,\mathrm{e}^{-\varepsilon^2\xi^2}\,\mathrm{d}\xi = \frac{\sqrt{\pi}}{\varepsilon}\exp\Big(-\frac{\pi^2(x-t)^2}{\varepsilon^2}\Big).$$

Therefore

$$\mathscr{I}_\varepsilon(x) = \int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\varepsilon}\exp\Big(-\frac{\pi^2(x-t)^2}{\varepsilon^2}\Big)f(t)\,\mathrm{d}t.$$

Letting $t = x+\varepsilon u$ we get

$$\mathscr{I}_\varepsilon(x) = \int_{-\infty}^{\infty}\sqrt{\pi}\,\mathrm{e}^{-\pi^2u^2}\,f(x+\varepsilon u)\,\mathrm{d}u.$$

Using the Gaussian integral (1.5.1) and the fact that f is continuous, we obtain

$$\mathscr{I}_\varepsilon(x)\xrightarrow[\varepsilon\to 0]{} f(x).$$

This together with (1.5.3) completes the proof. $\square$

1.5.1 The Mellin Transform

Definition 1.5.2. *Let $f\in L^1([0,\infty))$ and let*

$$D_f := \Big\{s\in\mathbb{C}\mid \int_0^{\infty} x^{s-1}f(x)\,\mathrm{d}x \textit{ exists}\Big\}.$$

We define the Mellin transform of f as the function $\mathcal{M}f\colon D_f\to\mathbb{C}$ defined by

$$\mathcal{M}f(s) = \int_0^{\infty} x^{s-1}f(x)\,\mathrm{d}x.$$

Now, let $\tilde{f} := f\circ\exp$. Then

$$\mathscr{F}\tilde{f}(\xi) = \int_{-\infty}^{\infty}\mathrm{e}^{-2\pi i\xi t}\,\tilde{f}(t)\,\mathrm{d}t.$$

Letting $x=e^t$, $s=-2\pi i\xi$ it is easy to see that

$$\mathscr{F}\tilde{f}(\xi) = \int_0^{\infty} x^{s-1}f(x)\,\mathrm{d}x = \mathcal{M}f(s).$$

Therefore, Fourier inversion (Theorem 1.5.4) immediately gives us

Theorem 1.5.5 (Mellin inversion theorem)**.** *Suppose that $f\colon[0,\infty)\to\mathbb{C}$ is a function such that its Mellin transform is well defined. Then*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\mathcal{M}f(s)}{x^s}\,\mathrm{d}s.$$

Chapter 2

Special Functions

2.1 The Gamma Function

This section is based on the book [10]. The *Gamma function* originated as a solution of an interpolation problem for the factorial function: *Find a smooth curve that connects the points* (x, y) *given by* $y = (x-1)!$ *at the positive integer values for* x. This problem can be solved by considering the so called *Euler's integral of the second kind*

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t}\, t^{z-1}\,\mathrm{d}t \qquad (\mathfrak{Re}(z) > 0),$$

in which the path of integration is the real axis and t^{z-1} takes its principal value. A graph of $|\Gamma(z)|$ for $z \in \mathbb{C}$ is shown in Figure 2.1.

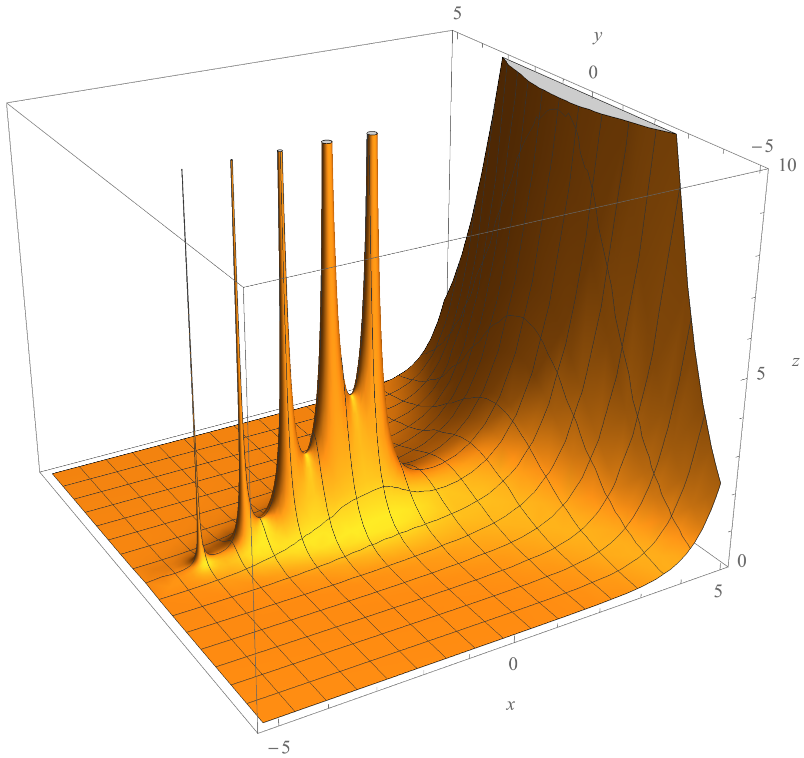

Figure 2.1: Absolute value of the complex Gamma function

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \Re(z) < \beta$. Then

$$|t^{z-1}| \leqslant t^{\alpha-1} \quad (0 < t \leqslant 1), \qquad |t^{z-1}| \leqslant t^{\beta-1} \quad (t \geqslant 1).$$

Therefore, by the Weierstrass' M-test, the integral converges uniformly with respect to z. That $\Gamma(z)$ is holomorphic in the half-plane $\mathfrak{Re}(z) > 0$ is a consequence of this result and Theorem 1.1.3. A single partial integration of the Gamma function yields the *fundamental recurrence formula*

$$\Gamma(z+1) = z\Gamma(z).$$

Hence if $z = n \in \mathbb{N}$ we obtain

$$\Gamma(n+1) = n!.$$

The recurrence formula enables Γ to be continued analytically into the left half-plane, except at the points $0, -1, -2, \ldots$. Now we use *Prym's decomposition* to answer the question about the nature of the non-positive integer points:

$$\Gamma(z) = \int_0^1 t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t + \int_1^\infty t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

The second integral represents an entire function of z. In the first integral we expand e^{-t} using it's Taylor series around $t = 0$ and then we use Theorem 1.1.1 to interchange summation and integration. This gives us the following expansion due to Mittag-Leffler

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)} + \int_1^\infty t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t,$$

which holds for all $z \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$. It follows that Γ has simple poles at each non-positive integer value. Moreover, for every $n \in \mathbb{N}$ we have

$$\lim_{z\to -n} (z+n)\Gamma(z) = \frac{(-1)^n}{n!}.$$

This means that the residue at the pole $-n$ equals $\frac{(-1)^n}{n!}$.

2.1.1 Euler's Limit Formula

An alternative definition of Γ, which is not restricted to the half-plane $\Re(z) > 0$, can be derived in the following way. We have

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n = \mathrm{e}^{-t}.$$

This leads us to consider the function

$$\Gamma_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,\mathrm{d}t \qquad (\mathfrak{Re}(z) > 0).$$

Repeated integration by parts of $\Gamma_n(z)$ with $g(t) = \left(1 - \frac{t}{n}\right)^n$ and $h(t) = t^{z-1}$ as in Theorem 1.1.2 yields

$$\Gamma_n(z) = \sum_{k=0}^{n-1} \frac{(n-1)\cdots(n-k+1)\left(1-\frac{t}{n}\right)^{n-k} t^{z+k}}{n^{k-1} z(z+1)\cdots(z+k)}\Bigg|_0^n + \int_0^n \frac{(n-1)\cdots 2\cdot 1\cdot t^{z+n-1}}{n^{n-1}z(z+1)\cdots(z+n-1)}\,\mathrm{d}t$$
$$= \frac{n^z n!}{z(z+1)\cdots(z+n)}.$$

Now we are going to prove that $\Gamma_n(z) \to \Gamma(z)$ as $n \to \infty$. Write

$$\Gamma(z) - \Gamma_n(z) = \mathscr{I}_1 + \mathscr{I}_2,$$

where

$$\mathscr{I}_1 = \int_n^\infty \mathrm{e}^{-t}\, t^{z-1}\,\mathrm{d}t \quad \text{and} \quad \mathscr{I}_2 = \int_0^n \left(\mathrm{e}^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,\mathrm{d}t.$$

Clearly $\mathscr{I}_1 \to 0$ as $n \to \infty$. When $0 \leqslant t < n$, for $\mathscr{I}_2$ we have,

$$\log\left(1-\frac{t}{n}\right)^n = n\log\left(1-\frac{t}{n}\right) = -t - \frac{t^2}{2n} - \frac{t^3}{3n^2} - \frac{t^4}{4n^3} + \cdots .$$

Hence

$$\left(1-\frac{t}{n}\right)^n = \exp\left\{-t - \left(\frac{t^2}{2n} + \frac{t^3}{3n^2} + \frac{t^4}{4n^3} + \cdots\right)\right\} = \mathrm{e}^{-t-A} \leqslant \mathrm{e}^{-t}, \tag{2.1.1}$$

where

$$A = \frac{t^2}{2n} + \frac{t^3}{3n^2} + \frac{t^4}{4n^3} + \cdots .$$

Note that if $t \leqslant \frac{n}{2}$ then $A \leqslant \frac{Bt^2}{n}$ where

$$B = \frac{1}{2} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 2^2} + \cdots$$

is a finite number. So,

$$\mathrm{e}^{-t} - \left(1-\frac{t}{n}\right)^n = \mathrm{e}^{-t}(1-\mathrm{e}^{-A}) \leqslant \mathrm{e}^{-t}A \leqslant \mathrm{e}^{-t}\frac{Bt^2}{n}. \tag{2.1.2}$$

Therefore, using (2.1.1) and (2.1.2) we obtain

$$|\mathscr{I}_2| \leqslant \left|\int_{n/2}^n \left(\mathrm{e}^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,\mathrm{d}t\right| + \left|\int_0^{n/2} \left(\mathrm{e}^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,\mathrm{d}t\right|$$

$$\leqslant \int_{n/2}^n 2\,\mathrm{e}^{-t}\, t^{\mathfrak{Re}(z)-1}\,\mathrm{d}t + \frac{B}{n}\int_0^{n/2} \mathrm{e}^{-t}\, t^{\mathfrak{Re}(z)+1}\,\mathrm{d}t$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This gives us the *Euler's limit formula*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}. \tag{2.1.3}$$

2.1.2 Infinite Product Formula

Now we are going to use Euler's limit formula to write $\Gamma(z)$ into the canonical form of an infinite product, we need the following lemma.

Lemma 2.1.1. *The sequence* $\{s_n\}$ *defined by*

$$s_n = \sum_{k=1}^{n} \frac{1}{k} - \log n$$

converges as $n \to \infty$.

Proof. Since $1/t$ is a decreasing function for $t > 0$, then for $n \geqslant 2$

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{\mathrm{d}t}{t} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

Therefore $1/n < s_n < 1$, so s_n is a sequence of positive numbers. Now,

$$s_{n+1} - s_n = \frac{1}{n+1} + \log\left(\frac{n}{n+1}\right) < 0.$$

This tells us that s_n is a decreasing sequence, thus we conclude that s_n converges as $n \to \infty$. □

The limiting value of s_n is called the *Euler-Mascheroni constant* and is usually denoted by γ. Numerical computations give

$$\gamma \approx 0.5772156649\ldots.$$

Let $z \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$. Note that

$$\frac{1}{\Gamma_n(z)} = z \exp\left\{z\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)\right\} \prod_{k=1}^{n} \left(\frac{z+k}{k}\right) \mathrm{e}^{-z/k}.$$

Letting $n \to \infty$, we obtain the required infinite product in the form

$$\frac{1}{\Gamma(z)} = z\,\mathrm{e}^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) \mathrm{e}^{-z/k}. \tag{2.1.4}$$

The left hand side function of (2.1.4) is called the *reciprocal Gamma function.*

2.1.3 The Reflection Formula

We will show in Corollary 3.4.1.1 that

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right).$$

Therefore, using (2.1.4) we obtain

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{-1}{z\Gamma(z)\Gamma(-z)} = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) = \frac{\sin \pi z}{\pi}.$$

This gives us the *Euler's reflection formula*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{2.1.5}$$

2.1.4 Gauss' Multiplication Formula

Let $m \in \mathbb{N} \setminus \{0\}$. For every z in the domain of $\Gamma(mz)$ and $\Gamma(z+\frac{k}{m})$ let

$$f(z) = \frac{m^{mz}}{\Gamma(mz)} \prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right).$$

Then, by means of the Euler's limit formula (2.1.3) we have

$$\begin{aligned} f(z) &= \lim_{n\to\infty} m^{mz} \frac{mz(mz+1)\cdots(mz+mn)}{(mn)!(mn)^{mz}} \times \frac{n!n^z}{z(z+1)\cdots(z+n)} \\ &\quad \times \frac{n!n^{z+\frac{1}{m}}}{(z+\frac{1}{m})\cdots(z+\frac{1}{m}+n)} \times \cdots \times \frac{n!n^{z+\frac{m-1}{m}}}{(z+\frac{m-1}{m})(z+\frac{m-1}{m}+n)} \\ &= \lim_{n\to\infty} \frac{m^{mn+1} n^{\frac{m-1}{2}} (n!)^{m-1}}{(mn)!}. \end{aligned}$$

The last quantity is independent of z, and must be finite since the left-hand side exists. This means that f is constant, particularly

$$f(z) = f\left(\frac{1}{m}\right) = m \prod_{k=0}^{m-1} \Gamma\left(\frac{k+1}{m}\right).$$

To evaluate the product, we rearrange it as follows

$$\prod_{k=0}^{m-1} \Gamma\left(\frac{k+1}{m}\right) = \Gamma\left(\frac{1}{m}\right)\Gamma\left(1-\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\Gamma\left(1-\frac{2}{m}\right)\cdots\Gamma\left(\frac{m-1}{2m}\right)\Gamma\left(1-\frac{m-1}{2m}\right).$$

Now we use the Euler's reflection formula (2.1.5) to obtain

$$\begin{aligned} \prod_{k=0}^{m-1} \Gamma\left(\frac{k+1}{m}\right) &= \left(\prod_{k=1}^{m-1} \frac{\pi}{\sin(\frac{\pi k}{m})}\right)^{1/2} \\ &= \left(\prod_{k=1}^{m-1} \frac{2\pi i\, \mathrm{e}^{\frac{i\pi k}{m}}}{\mathrm{e}^{\frac{2\pi i k}{m}} - 1}\right)^{1/2} \\ &= (2\pi)^{\frac{m-1}{2}} \mathrm{e}^{\frac{i\pi(m-1)}{2}} \left(\prod_{k=1}^{m-1} \frac{1}{\xi_k - 1}\right)^{1/2}, \end{aligned}$$

where $\xi_k = \mathrm{e}^{\frac{2\pi i k}{m}}$. Note that

$$x^m - 1 = (x-\xi_1)(x-\xi_2)\cdots(x-\xi_{m-1})(x-1).$$

Therefore

$$(x-\xi_1)(x-\xi_2)\cdots(x-\xi_{m-1}) = 1 + x + \cdots x^{m-1}.$$

In particular, for $x = 1$ we obtain

$$\prod_{k=1}^{m-1} (\xi_k - 1) = \mathrm{e}^{i\pi(m-1)}\, m.$$

Thus

$$f(z) = m^{\frac{1}{2}} (2\pi)^{\frac{m-1}{2}}.$$

This gives us the *Gauss' multiplication formula*

$$\prod_{k=0}^{m-1} \Gamma\left(z+\frac{k}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz} \Gamma(mz). \tag{2.1.6}$$

The special case $m=2$ reduces to

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \tag{2.1.7}$$

and is called the *Legendre's duplication formula*

2.1.5 Hankel's Loop Integral

The final formula that we will need for the Gamma function is an integral representation that is constructed by using a loop contour in the complex plane. The idea is due to Hankel (1864).

Consider

$$I(z) = \int_\gamma \mathrm{e}^t\, t^{-z}\, \mathrm{d}t,$$

where the path γ begins at $t=-\infty$, encircles $t=0$ once in the positive orientation, and returns to its starting point;

We suppose that the branch of t^{-z} takes its principal value at the point where the contour crosses the positive real axis, and is continuous elsewhere. Let $\varepsilon > 0$. Then by Cauchy's integral theorem the path can be deformed into the two sides of the interval $(-\infty, -\varepsilon]$, together with the circle $|t| = \varepsilon$; see Figure 2.2.

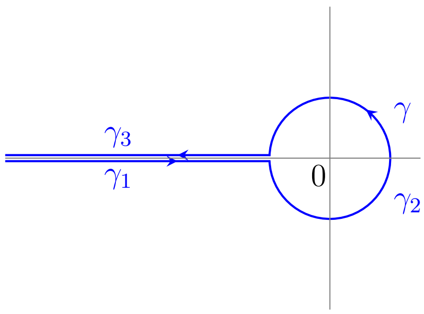

Figure 2.2: The loop contour γ in the complex plane.

Thus

$$I(z) = \int_{\gamma_1} \mathrm{e}^t\, t^{-z}\, \mathrm{d}t + \int_{\gamma_2} \mathrm{e}^t\, t^{-z}\, \mathrm{d}t + \int_{\gamma_3} \mathrm{e}^t\, t^{-z}\, \mathrm{d}t,$$

where γ_1 is the lower side of the negative real axis (here $\arg t = -\pi$), γ_2 is the circle $|t| = \varepsilon$ and γ_3 is the upper side of the negative real axis (here $\arg t = \pi$). Assume temporarily that $\mathfrak{Re}(z) < 1$. Then

$$\int_{\gamma_2} \mathrm{e}^t\, t^{-z}\, \mathrm{d}t = \int_0^{2\pi} \varepsilon i\, \mathrm{e}^{\varepsilon(\cos\theta + i\sin\theta)} (\varepsilon\, \mathrm{e}^{i\theta})^{-z}\, \mathrm{e}^{i\theta}\, \mathrm{d}\theta \xrightarrow[\varepsilon\to 0]{} 0.$$

For γ_1 and γ_3, write $\tau = |t|$, and we get

$$I(z) = -\int_\infty^0 \mathrm{e}^{-\tau}\, \tau^{-z}\, \mathrm{e}^{i\pi z}\, \mathrm{d}\tau - \int_0^\infty \mathrm{e}^{-\tau}\, \tau^{-z}\, \mathrm{e}^{-i\pi z}\, \mathrm{d}\tau = 2i\sin(\pi z)\Gamma(1-z) = \frac{2\pi i}{\Gamma(z)}.$$

Therefore

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\gamma} \mathrm{e}^{t}\, t^{-z}\,\mathrm{d}t. \tag{2.1.8}$$

This is *Hankel's loop integral*. Analytic continuation removes the temporary restriction on $\mathfrak{Re}(z)$, provided that the branch of t^{-z} is chosen as discussed before. Now, with the aid of Theorem 1.1.3 we see that the right hand side of (2.1.8) represents an entire function. This shows that the reciprocal Gamma function is an entire function. As a corollary, the Gamma function has no zeros. The graph of $\Gamma(x)$ and $\frac{1}{\Gamma(x)}$, for $x \in \mathbb{R}$ is shown in Figure 2.3

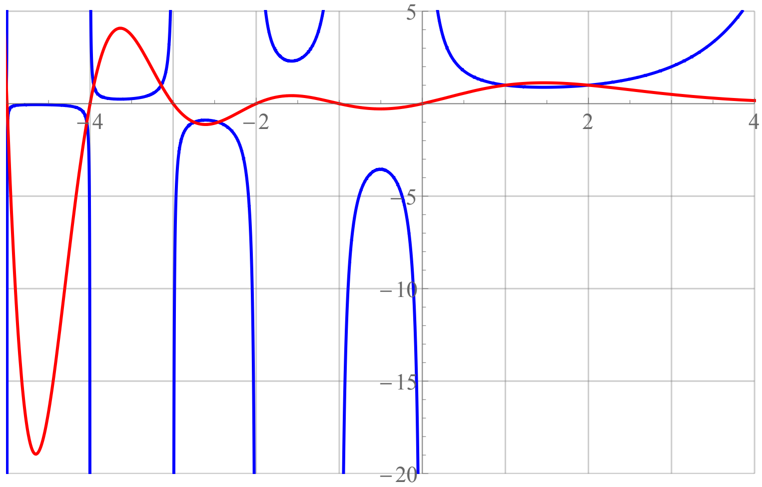

Figure 2.3: The Gamma function in blue and the reciprocal Gamma function in red.

2.1.6 The Bohr-Mollerup Theorem

The problem of finding a continuous function of $x > 0$ that equaled $(n-1)!$ at $x = n \in \mathbb{N}$ originated the Gamma function, but clearly it is not the unique solution to this problem. The condition of convexity is not enough, but the fact that the Gamma function appears so frequently suggests that it is unique in some sense. The conditions for uniqueness were found by Bohr and Mollerup in 1922.

Definition 2.1.1. *A function $f\colon (a,b) \to \mathbb{R}$ is convex if for all $x, y \in (a,b)$ and for every $\alpha \in (0,1)$,*

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y).$$

Geometrically, this means that the line segment between any two points on the graph of the function lies above or on the graph.

If $f\colon (a,b) \to \mathbb{R}$ is a positive function such that $\log f$ is convex, then we say that f is *logarithmically convex*. Note that if $f\colon (a,b) \to \mathbb{R}$ is convex and $a < x < y < z < b$, then

$$\frac{f(y) - f(x)}{y - x} \leqslant \frac{f(z) - f(x)}{z - x}. \tag{2.1.9}$$

Theorem 2.1.1. *Suppose that $f\colon (0,\infty) \to \mathbb{R}$ is a function such that*

(i) $f(1) = 1$

(ii) $f(x+1) = x f(x)$

(iii) f is logarithmically convex.

Then $f(x) = \Gamma(x)$.

Proof. Let $n \in \mathbb{N}$ and let $x \in (0,1)$. Note that conditions (i) and (ii) imply that it is sufficient to prove the theorem for such x. Since f is logarithmically convex and $n < n+1 < n+1+x < n+2$, we use (2.1.9) to obtain

$$\log \frac{f(n+1)}{f(n)} \leqslant \frac{1}{x} \log \frac{f(n+1+x)}{f(n+1)} \leqslant \log \frac{f(n+2)}{f(n+1)}.$$

Using conditions (i) and (ii) we may simplify this inequalities to get

$$x \log n \leqslant \log \frac{(x+n)(x+n-1)\cdots x f(x)}{n!} \leqslant x \log(n+1).$$

Substracting $x \log n$ we may rearrange the inequalities as follows:

$$0 \leqslant \log \frac{x(x+1)\cdots(x+n)}{n! n^x} + \log f(x) \leqslant x \log\left(1 + \frac{1}{n}\right).$$

Therefore,

$$f(x) = \lim_{n \to \infty} \frac{n! n^x}{x(x+1)\cdots(x+n)} = \Gamma(x).$$ □

2.2 The Digamma Function

This section is based on [12]. The *digamma function* is defined as the logarithmic derivative of the Gamma function and is usually denoted by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

A graph of $\psi(x)$ for $x \in \mathbb{R}$ is shown in Figure 2.4. By using the infinite product (2.1.4) we obtain

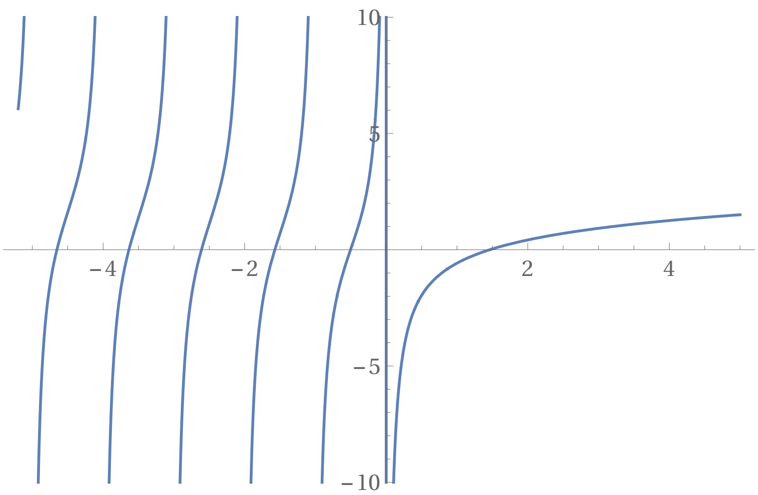

Figure 2.4: The digamma function

$$\log\Gamma(z) = -\log z - \gamma z + \sum_{n=1}^{\infty} \frac{z}{n} - \log\left(1+\frac{z}{n}\right), \quad z \neq 0, -1, -2, \ldots.$$

Hence

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z}\right), \quad z \neq 0, -1, -2, \ldots. \tag{2.2.1}$$

It follows that the digamma function possesses simple poles at all non positive integers. Using (2.2.1) we obtain the recurrence formula for the ψ function:

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \tag{2.2.2}$$

Special values at positive integers follow from the series in (2.2.1):

$$\psi(1) = -\gamma, \quad \psi(m) = -\gamma + \sum_{n=1}^{m-1} \frac{1}{n} \quad (m = 2, 3, \ldots). \tag{2.2.3}$$

Now we are going to find an integral representation for ψ that will be used to find an asymptotic expansion for $\log\Gamma$. We first establish two lemmas.

Lemma 2.2.1. *The Euler-Mascheroni constant γ has the following integral representations:*

$$-\int_0^{\infty} \mathrm{e}^{-t}\log t\,\mathrm{d}t, \quad \int_0^1 \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t - \int_1^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t, \quad \int_0^{\infty} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} - \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

Proof. The first representation follows from

$$\gamma = -\psi(1) = -\Gamma'(1) = -\int_0^{\infty} \mathrm{e}^{-t}\log t\,\mathrm{d}t.$$

The second one follows from

$$\begin{aligned}\int_1^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t &= \int_1^{\infty} \mathrm{e}^{-t}\log t\,\mathrm{d}t \\ &= \int_0^{\infty} \mathrm{e}^{-t}\log t\,\mathrm{d}t - \int_0^1 \mathrm{e}^{-t}\log t\,\mathrm{d}t \\ &= -\gamma - \int_0^1 \mathrm{e}^{-t}\log t\,\mathrm{d}t.\end{aligned}$$

Hence

$$\gamma = -\int_0^1 \mathrm{e}^{-t}\log t\,\mathrm{d}t - \int_1^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t = \int_0^1 \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t - \int_1^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

For the last one, write $t = 1 - \mathrm{e}^{-u}$ to get

$$\begin{aligned}\gamma = \int_0^1 \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t - \int_1^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t &= \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\infty} \frac{\mathrm{d}t}{t} - \int_{\varepsilon}^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\infty} \frac{\mathrm{e}^{-u}}{1-\mathrm{e}^{-u}}\,\mathrm{d}u - \int_{\varepsilon}^{\infty} \frac{\mathrm{e}^{-u}}{u}\,\mathrm{d}u \\ &= \int_0^{\infty} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} - \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.\end{aligned}$$

□

Lemma 2.2.2. *For every $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, we have*

$$\psi(z) = \int_0^\infty \frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-zt}}{1 - \mathrm{e}^{-t}} \, \mathrm{d}t \quad (\mathfrak{Re}(z) > 0).$$

Proof. For every $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ we know that

$$\frac{1}{n+z} = \int_0^\infty \mathrm{e}^{-t(n+z)} \, \mathrm{d}t \quad (\mathfrak{Re}(z) > 0).$$

Therefore, using (2.2.1) and Lemma 2.2.1 we have

$$\begin{aligned}
\psi(z) &= -\gamma + \sum_{n=0}^\infty \frac{1}{n+1} - \frac{1}{n+z} \\
&= \int_0^\infty \frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}} \, \mathrm{d}t + \sum_{n=0}^\infty \int_0^\infty \mathrm{e}^{-t(n+1)} \, \mathrm{d}t - \int_0^\infty \mathrm{e}^{-t(n+z)} \, \mathrm{d}t.
\end{aligned}$$

When $\mathfrak{Re}(z) > 0$, we use Theorem 1.1.1 to interchange summation and integration. This gives

$$\begin{aligned}
\psi(z) &= \int_0^\infty \left(\frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}} + \sum_{n=0}^\infty \mathrm{e}^{-t(n+1)} - \mathrm{e}^{-t(n+z)} \right) \mathrm{d}t \\
&= \int_0^\infty \frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-zt}}{1 - \mathrm{e}^{-t}} \, \mathrm{d}t.
\end{aligned}$$

□

Now, let $z \in \mathbb{C}$ and consider the integral

$$F(z) = \int_0^\infty \frac{\mathrm{e}^{-t} - \mathrm{e}^{-zt}}{t} \, \mathrm{d}t \quad (\mathfrak{Re}(z) > 0).$$

Then, by Theorem 1.1.3, F is holomorphic and we may differentiate under the integral sign to obtain

$$F'(z) = \int_0^\infty \mathrm{e}^{-zt} \, \mathrm{d}t = \frac{1}{z}.$$

Note that

$$F'(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log z \quad \text{and} \quad F(1) = \log 1,$$

where $\log z$ denotes the principal branch of the logarithm. This shows that

$$\int_0^\infty \frac{\mathrm{e}^{-t} - \mathrm{e}^{-zt}}{t} \, \mathrm{d}t = \log z \quad (\mathfrak{Re}(z) > 0). \tag{2.2.4}$$

Now we are ready to give an integral representation for ψ due to Binet (1839).

Theorem 2.2.1. *For every $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, we have*

$$\psi(z+1) = \log z + \frac{1}{2z} - \int_0^\infty t g(t) \, \mathrm{e}^{-zt} \, \mathrm{d}t \quad (\mathfrak{Re}(z) > 0), \tag{2.2.5}$$

where

$$g(t) = \frac{1}{t} \left(\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \frac{1}{2} \right) = \sum_{n=1}^\infty \frac{B_{2n}}{(2n)!} t^{2n-2} \quad (|t| < 2\pi). \tag{2.2.6}$$

Proof. We use Lemma 2.2.2 and (2.2.4) to get

$$\begin{aligned}\psi(z+1) &= \int_0^\infty \frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-t(z+1)}}{1-\mathrm{e}^{-t}}\,\mathrm{d}t\\ &= \int_0^\infty \frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-zt}}{\mathrm{e}^{t}-1}\,\mathrm{d}t\\ &= \log z + \int_0^\infty \mathrm{e}^{-zt}\left(\frac{1}{t} - \frac{1}{\mathrm{e}^{t}-1}\right)\mathrm{d}t\\ &= \log z + \frac{1}{2z} - \int_0^\infty \mathrm{e}^{-zt}\left(\frac{1}{\mathrm{e}^{t}-1} - \frac{1}{t} + \frac{1}{2}\right)\mathrm{d}t,\end{aligned}$$

which is equivalent to (2.2.5). The series expansion for g follows from the fact that

$$\frac{t}{\mathrm{e}^{t}-1} - 1 + \frac{t}{2} = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} t^{2n}.$$

This completes the proof. □

Now we are ready to find an asymptotic expansion for $\log\Gamma$. Integrating (2.2.5) we obtain

$$\log\Gamma(z+1) = \left(z+\tfrac{1}{2}\right)\log z - z + \int_0^\infty g(t)\,\mathrm{e}^{-zt}\,\mathrm{d}t + C, \tag{2.2.7}$$

where C is a constant of integration, which has to be determined. Since

$$g(t) = \frac{1}{t}\left(\frac{1}{\mathrm{e}^{t}-1} - \frac{1}{t} + \frac{1}{2}\right) \quad \text{and} \quad g(t) = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} t^{2n-2} = \frac{1}{12} + \mathcal{O}(t) \quad (t \to 0),$$

then

$$\lim_{t\to\infty} g(t) = 0 \quad \text{and} \quad \lim_{t\to 0} g(t) = \frac{1}{12}.$$

Moreover, g is decreasing for $t > 0$, since

$$g'(t) = \frac{-2\,\mathrm{e}^{t}(t^2+4) - \mathrm{e}^{2t}(t-4) + t + 4}{2t^3(\mathrm{e}^{t}-1)^2} < 0 \quad (t > 0).$$

This shows that g is bounded for $t > 0$. Thus

$$\int_0^\infty g(t)\,\mathrm{e}^{-zt}\,\mathrm{d}t = o(1) \quad (z \to \infty). \tag{2.2.8}$$

From (2.2.7) we also get

$$\log\Gamma\left(z+\tfrac{1}{2}\right) = z\log\left(z-\tfrac{1}{2}\right) - \left(z-\tfrac{1}{2}\right) + \int_0^\infty g(t)\,\mathrm{e}^{-(z-\frac{1}{2})t}\,\mathrm{d}t + C, \tag{2.2.9}$$

and

$$\log\Gamma(2z+1) = \left(2z+\tfrac{1}{2}\right)\log 2z - 2z + \int_0^\infty g(t)\,\mathrm{e}^{-2zt}\,\mathrm{d}t + C. \tag{2.2.10}$$

Now define the following functions:

$$h(z) = \left(z+\tfrac{1}{2}\right)\log z - z \quad \text{and} \quad H(z) = h(z) + h\left(z-\tfrac{1}{2}\right) - h(2z).$$

It is easy to see that

$$H(z) = z \log \frac{2z-1}{8z} + \frac{1}{2} - \frac{1}{2}\log 2.$$

Expanding H in power series about infinity [1] we get

$$H(z) = -2z\log 2 - \frac{1}{2}\log 2 + \mathcal{O}\Big(\frac{1}{z}\Big) \quad (z\to\infty).$$

Combining (2.2.7), (2.2.9), (2.2.10), and writing the integrals as in (2.2.8) we get

$$\log\frac{\Gamma(z+1)\Gamma(z+\frac{1}{2})}{\Gamma(2z+1)} = H(z) + o(1) + C. \quad (z\to\infty) \tag{2.2.11}$$

The Legendre's duplication formula (2.1.7) can be written in the form

$$\log\frac{2^{2z}\Gamma(z+1)\Gamma(z+\frac{1}{2})}{\Gamma(2z+1)} = \frac{1}{2}\log\pi.$$

Thus, (2.2.11) becomes

$$\frac{1}{2}\log\pi - \log 2^{2z} = -2z\log 2 - \frac{1}{2}\log 2 + \mathcal{O}\Big(\frac{1}{z}\Big) + o(1) + C \quad (z\to\infty).$$

Letting $z\to\infty$ we obtain $C = \frac{1}{2}\log 2\pi$. Since

$$\log\Gamma(z+1) = \log z + \log\Gamma(z),$$

we can write (2.2.7) in the form

$$\log\Gamma(z) = \log\Big(\sqrt{2\pi}z^{z-\frac{1}{2}}\,\mathrm{e}^{-z}\Big) + \int_0^\infty g(t)\,\mathrm{e}^{-zt}\,\mathrm{d}t.$$

Using the series expansion for g (2.2.6), we find via Watson's lemma (Theorem 1.4.2) an asymptotic expansion for the logarithm of the gamma function (*Stirling's series*):

$$\log\Gamma(z) \sim \log\Big(\sqrt{2\pi}z^{z-\frac{1}{2}}\,\mathrm{e}^{-z}\Big) + \sum_{n=1}^\infty \frac{B_{2n}}{2n(2n-1)}\frac{1}{z^{2n-1}}, \quad (z\to\infty). \tag{2.2.12}$$

Since the singularities of g are located on the imaginary axis, the above expansion holds for $|\arg z| < \pi$. Taking the exponential of this result, we get the generalization of *Stirling's formula*:

$$\Gamma(z) \sim \sqrt{2\pi}z^{z-\frac{1}{2}}\,\mathrm{e}^{-z}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \cdots\right), \quad (z\to\infty). \tag{2.2.13}$$

2.3 The Exponential, Logarithmic, Sine, and Cosine Integrals

2.3.1 The Exponential and Logarithmic Integral

This section is based on [10]. The *exponential integral* is defined for $z\in\mathbb{C}$ by the formula

$$\mathrm{E}_1(z) = \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

[1]We say that $f(z)$ is analytic at $z=\infty$ if the function $g(w) = f(1/w)$ is analytic at $w=0$. Thus we let $w = 1/z$, and consider the behavior of $f(z)$ at $z=\infty$ by studying the behavior of $g(w)$ at $w=0$.

Since $t = 0$ is a pole of the integrand, $z = 0$ is a branch point of $\mathrm{E}_1(z)$. The principal branch is obtained by introducing a cut along the negative real axis. Now suppose $z > 0$, letting $t = (1+u)z$ we obtain

$$\mathrm{E}_1(z) = \mathrm{e}^{-z} \int_0^\infty \frac{\mathrm{e}^{-uz}}{1+u}\,\mathrm{d}u.$$

Since the above integral converges if $|\arg z| < \frac{\pi}{2}$, then, by analytic continuation we extend the result for complex z to obtain

$$\mathrm{E}_1(z) = \mathrm{e}^{-z} \int_0^\infty \frac{\mathrm{e}^{-tz}}{1+t}\,\mathrm{d}t, \quad |\arg z| < \frac{\pi}{2}.$$

The *complementary exponential integral* is defined by

$$\mathrm{Ein}(z) = \int_0^z \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t,$$

and is entire. By expanding the integrand using it's Taylor series around $t = 0$ and then integrating the resulting series term by term with the aid of Theorem 1.1.1 we get

$$\mathrm{Ein}(z) = \int_0^z \sum_{n=0}^\infty \frac{(-1)^{n+1}t^n}{(n+1)!}\,\mathrm{d}t = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\frac{z^n}{n!}.$$

Now suppose temporarily that $z > 0$. Then

$$\begin{aligned}
\mathrm{Ein}(z) &= \int_0^1 \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t + \int_1^\infty \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t - \int_z^\infty \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t \\
&= \int_0^1 \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t + \int_1^z \frac{\mathrm{d}t}{t} - \int_1^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t + \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \\
&= \int_0^1 \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t - \int_1^\infty \frac{\mathrm{e}^{-t}}{t} + \log z + \mathrm{E}_1(z).
\end{aligned}$$

Using Lemma 2.2.1 we obtain

$$\mathrm{Ein}(z) = \gamma + \log z + \mathrm{E}_1(z), \tag{2.3.1}$$

and again analytic continuation extends this result for complex z. When $z = x \in \mathbb{R}$, another notation used for the exponential integral is given by

$$\mathrm{Ei}(x) = ⨍_{-\infty}^x \frac{\mathrm{e}^t}{t}\,\mathrm{d}t \quad (x \neq 0), \tag{2.3.2}$$

where $⨍$ means that the integral takes its Cauchy principal value when x is positive[2]. The connection with the previous notation is

$$\mathrm{E}_1(x) = -\mathrm{Ei}(-x), \quad \mathrm{E}_1(-x \pm 0i) = -\mathrm{Ei}(x) \mp i\pi.$$

The first identity follows from from the substitution $t = -u$. For the second one, use (2.3.1) to get

$$\begin{aligned}
\mathrm{E}_1(-x \pm 0i) &= -\gamma - \log(-x \pm 0i) + \mathrm{Ein}(-x \pm 0i) \\
&= -\gamma - \log(-x) \mp i\pi + \mathrm{Ein}(-x) \\
&= -\mathrm{Ei}(-x) \mp i\pi.
\end{aligned}$$

[2]That is, $\lim_{\epsilon \to 0^+}(\int_{-\infty}^{-\epsilon} + \int_\epsilon^x)$.

A related function is the *logarithmic integral* defined for $x > 0$ as

$$\operatorname{li}(x) = \int_0^x \frac{\mathrm{d}t}{\log t} \quad (x \neq 1). \tag{2.3.3}$$

The substitution $t = \log u$ in (2.3.2) gives us

$$\operatorname{Ei}(\log x) = \operatorname{li}(x) \quad (x \neq 1).$$

To avoid the singularity of the integrand at $t = 1$ in (2.3.3) we define the *offset logarithmic integral* or *Eulerian logarithmic integral* as

$$\operatorname{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}.$$

The relation with the logarithmic integral is

$$\operatorname{Li}(x) = \operatorname{li}(x) - \operatorname{li}(2).$$

2.3.2 The Sine and Cosine Integrals

The *sine integrals* are defined by

$$\operatorname{Si}(z) = \int_0^z \frac{\sin t}{t}\,\mathrm{d}t, \quad \operatorname{si}(z) = -\int_z^\infty \frac{\sin t}{t}\,\mathrm{d}t,$$

and each function is entire. To relate them we need the following lemma:

Lemma 2.3.1. *We have*

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}.$$

Proof. Let $\mathscr{C}$ be the contour of integration shown in Figure 2.5

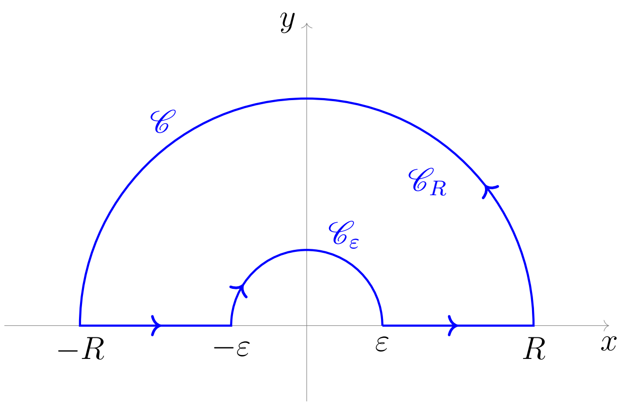

Figure 2.5: Contour of integration $\mathscr{C}$

Then, by Cauchy's integral theorem we have

$$\int_{\mathscr{C}} \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t = 0.$$

This shows that

$$\int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t + \int_\varepsilon^R \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t = -\int_{\mathscr{C}_\varepsilon} \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t - \int_{\mathscr{C}_R} \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t. \tag{2.3.4}$$

On the small semicircle $t = \epsilon\,\mathrm{e}^{i\theta}$, $\pi \geqslant \theta \geqslant 0$. Therefore

$$\int_{\mathscr{C}_\varepsilon} \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t = \int_\pi^0 i e^{i\varepsilon\,\mathrm{e}^{i\theta}}\,\mathrm{d}\theta \xrightarrow[\varepsilon\to 0]{} -i\pi.$$

On the large semicircle $t = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$. Thus

$$\left|\int_{\mathscr{C}_R} \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t\right| \leqslant \int_0^\pi \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta \leqslant 2\int_0^{\frac{\pi}{2}} \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta.$$

Now we use *Jordan's inequality*[3]:

$$\sin\theta \geqslant \frac{2}{\pi}\theta \qquad (0 \leqslant \theta \leqslant \frac{\pi}{2}).$$

This gives us

$$\left|\int_{\mathscr{C}_R} \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t\right| \leqslant 2\int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{2}{\pi}R\theta}\,\mathrm{d}\theta = \frac{\pi}{R}(1-\mathrm{e}^{-R}) \xrightarrow[R\to\infty]{} 0.$$

Letting $\varepsilon \to 0$ and $R \to \infty$ in (2.3.4) we obtain

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t = i\pi.$$

Taking imaginary part and using the fact that $\dfrac{\sin t}{t}$ is an even function the result follows. $\square$

Using the definition of the sine integrals and the previous lemma we get

$$\mathrm{Si}(z) = \frac{\pi}{2} + \mathrm{si}(z).$$

The *cosine integrals* are defined by

$$\mathrm{Ci}(z) = -\int_z^\infty \frac{\cos t}{t}\,\mathrm{d}t, \quad \mathrm{Cin}(z) = \int_0^z \frac{1-\cos t}{t}\,\mathrm{d}t.$$

Ci has a branch point at $z = 0$ and the principal branch is obtained by introducing a cut along the negative real axis. Cin is entire. By a change of variables we can express the sine and cosine integrals in terms of the exponential integrals, the result is

$$2i\,\mathrm{Si}(z) = \mathrm{Ein}(iz) - \mathrm{Ein}(-iz), \qquad 2i\,\mathrm{si}(z) = \mathrm{E}_1(iz) - \mathrm{E}_1(-iz),$$

and

$$2\,\mathrm{Ci}(z) = -\,\mathrm{E}_1(iz) - \mathrm{E}_1(-iz), \qquad 2\,\mathrm{Cin}(z) = \mathrm{Ein}(iz) + \mathrm{Ein}(-iz).$$

This result and (2.3.1) connects the two cosine integrals:

$$\mathrm{Ci}(z) + \mathrm{Cin}(z) = \log z + \gamma.$$

2.4 Arithmetic Functions

Definition 2.4.1. *An arithmetic function is a map $f : A \subseteq \mathbb{N} \to \mathbb{C}$.*

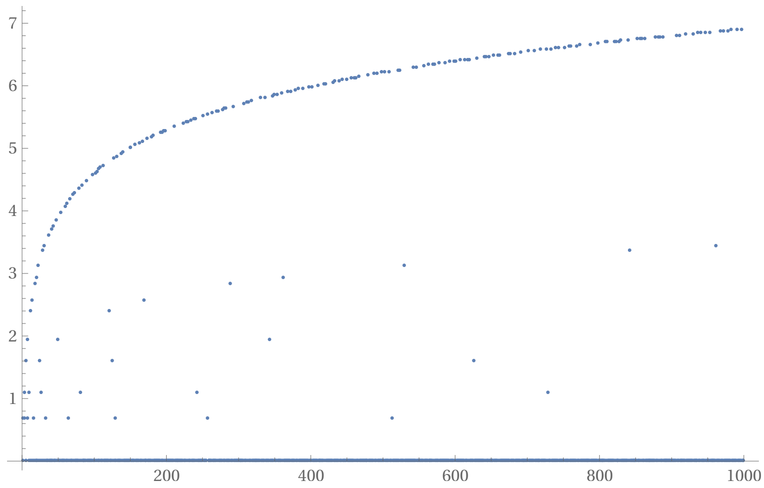

Figure 2.6: The Mangoldt function

For $n \in \mathbb{N}$ we define the *Mangoldt function* as

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some } p \in \mathbb{P} \text{ and for some } m \in \mathbb{N}^*; \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.1}$$

A graph of this function is shown in Figure 2.6.

Theorem 2.4.1. *For $n \in \mathbb{N}^*$ we have*

$$\log n = \sum_{d \mid n} \Lambda(d).$$

Proof. The proof runs by induction on n. For $n = 1$, clearly $\log 1 = \Lambda(1)$. Now suppose $n > 1$ and write

$$n = \prod_{j=1}^{k} p_j^{\alpha_j},$$

where $p_1 < p_2 < \cdots < p_k$ are primes and the α_j are positive integers. Therefore

$$\sum_{d \mid n} \Lambda(d) = \sum_{j=1}^{k} \sum_{m=1}^{\alpha_j} \Lambda(p_j^m) = \sum_{j=1}^{k} \alpha_j \log p_j = \log n. \qquad \square$$

For $x > 0$ we define the *prime counting function* π as the number of primes less than or equal to x, that is

$$\pi(x) = \sum_{p \leqslant x} 1.$$

The *Chebyshev's ψ and ϑ functions* are defined as[4]

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n), \quad \text{and} \quad \vartheta(x) = \sum_{p \leqslant x} \log p.$$

[3]This inequality easily follows by comparing the graph of the functions $\sin x$ and $\frac{2}{\pi}x$ on the interval $(0, \frac{\pi}{2})$.

[4]Do not confuse with the digamma function which is also denoted by ψ. To avoid all possible confusions, some writers use $\psi^{(0)}$ for the digamma function and confine ψ for the Chebyshev function.

A graph of the Chebyshev's ψ function is shown in Figure 2.7. Now,

$$\psi(x) = \sum_{n\leqslant x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p^m \leqslant x} \Lambda(p^m) = \sum_{m=1}^{\infty} \sum_{p\leqslant x^{\frac{1}{m}}} \log p.$$

Note that the sum over m is finite because the sum on p is empty if $x^{\frac{1}{m}} < 2$ or equivalently if $m > \log_2 x$. Therefore,

$$\psi(x) = \sum_{m\leqslant \log_2 x} \sum_{p\leqslant x^{\frac{1}{m}}} \log p = \sum_{m\leqslant \log_2 x} \vartheta(x^{\frac{1}{m}}). \tag{2.4.2}$$

Theorem 2.4.2. *For $x > 0$ we have*

$$0 \leqslant \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \leqslant \frac{\log^2 x}{\sqrt{x}\log 4}.$$

Proof. From (2.4.2) we obtain

$$\psi(x) = \sum_{2\leqslant m\leqslant \log_2 x} \vartheta(x^{\frac{1}{m}}) + \vartheta(x).$$

Thus

$$\psi(x) - \vartheta(x) = \sum_{2\leqslant m\leqslant \log_2 x} \vartheta(x^{\frac{1}{m}}) \geqslant 0.$$

From the definition of ϑ we have

$$\vartheta(x) = \sum_{p\leqslant x} \log p \leqslant \sum_{p\leqslant x} \log x \leqslant x \log x.$$

Therefore

$$0 \leqslant \psi(x) - \vartheta(x) \leqslant \sum_{2\leqslant m\leqslant \log_2 x} x^{\frac{1}{m}} \log x^{\frac{1}{m}} \leqslant \log_2 x \sqrt{x} \log \sqrt{x} = \frac{\sqrt{x}\log^2 x}{\log 4},$$

and the result follows. $\square$

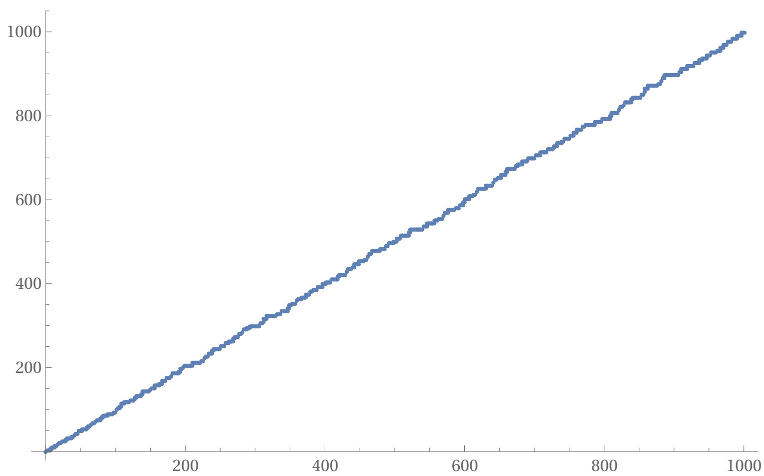

Figure 2.7: The Chebyshev's ψ function

Lemma 2.4.1 (Abel's identity)**.** *Given an arithmetic function a, let $A\colon \mathbb{R} \to \mathbb{C}$ be defined as*

$$A(t) = \begin{cases} \sum_{n \leqslant t} a(n), & \text{if } t \geqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varphi \in C^1[y, x]$, where $0 < y < x$. Then

$$\sum_{y<n\leqslant x} a(n)\varphi(n) = A(x)\varphi(x) - A(y)\varphi(y) - \int_y^x A(t)\varphi'(t)\,\mathrm{d}t. \tag{2.4.3}$$

Proof. Note that the sum in (2.4.3) can be expressed as a Riemann-Stieltjes integral:

$$\sum_{y<n\leqslant x} a(n)\varphi(n) = \int_y^x \varphi(t)\,\mathrm{d}A(t).$$

Integrating by parts and using the fact that $\varphi \in C^1[y, x]$ we obtain

$$\begin{aligned} \sum_{y<n\leqslant x} a(n)\varphi(n) &= A(x)\varphi(x) - A(y)\varphi(y) - \int_y^x A(t)\,\mathrm{d}\varphi(t) \\ &= A(x)\varphi(x) - A(y)\varphi(y) - \int_y^x A(t)\varphi'(t)\,\mathrm{d}t. \end{aligned}$$

□

Theorem 2.4.3. *For $x \geqslant 2$ we have*

$$\vartheta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,\mathrm{d}t,$$

and

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t\log^2 t}\,\mathrm{d}t.$$

Proof. Let $a : \mathbb{N} \to \{0, 1\}$ be the characteristic function of the primes, that is

$$a(n) = \begin{cases} 1, & \text{if } n \in \mathbb{P}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\pi(x) = \sum_{p\leqslant x} 1 = \sum_{1<n\leqslant x} a(n), \quad \text{and} \quad \vartheta(x) = \sum_{p\leqslant x} \log p = \sum_{1<n\leqslant x} a(n)\log n.$$

Using Lemma 2.4.1 with $\varphi(t) = \log t$ and $y = 1$ we get

$$\begin{aligned} \vartheta(x) = \sum_{1<n\leqslant x} a(n)\log n &= \pi(x)\log x - \pi(1)\log 1 - \int_1^x \frac{\pi(t)}{t}\,\mathrm{d}t \\ &= \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,\mathrm{d}t. \end{aligned}$$

Now, let $b(n) = a(n)\log n$. Then

$$\pi(x) = \sum_{\sqrt{2}<n\leqslant x} b(n)\frac{1}{\log n}, \quad \text{and} \quad \vartheta(x) = \sum_{n\leqslant x} b(n).$$

Again, we use Lemma 2.4.1 with $\varphi(t) = \dfrac{1}{\log t}$ and $y = \sqrt{2}$ to obtain

$$\begin{aligned}\pi(x) &= \frac{\vartheta(x)}{x} - \frac{\vartheta(\sqrt{2})}{\log\sqrt{2}} + \int_{\sqrt{2}}^{x} \frac{\vartheta(t)}{t\log^2 t}\,\mathrm{d}t \\ &= \frac{\vartheta(x)}{\log x} + \int_{2}^{x} \frac{\vartheta(t)}{t\log^2 t}\,\mathrm{d}t.\end{aligned}$$

This completes the proof. □

Theorem 2.4.4. *The following assertions are equivalent:*

(1) $\pi(x) \sim \dfrac{x}{\log x} \quad (x \to \infty)$

(2) $\vartheta(x) \sim x \quad (x \to \infty)$

(3) $\psi(x) \sim x \quad (x \to \infty)$

Proof. (2) $\iff$ (3) follows from Theorem 2.4.2, so it suffices to show (1) $\iff$ (2). First suppose that

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to \infty).$$

This implies that

$$\frac{1}{x}\int_{2}^{x} \frac{\pi(t)}{t}\,\mathrm{d}t = \mathcal{O}\Big(\frac{1}{x}\int_{2}^{x} \frac{\mathrm{d}t}{\log t}\Big).$$

Now,

$$\int_{2}^{x} \frac{\mathrm{d}t}{\log t} = \int_{2}^{\sqrt{x}} \frac{\mathrm{d}t}{\log t} + \int_{\sqrt{x}}^{x} \frac{\mathrm{d}t}{\log t} \leqslant \frac{\sqrt{x}-2}{\log 2} + \frac{x-\sqrt{x}}{\log\sqrt{x}}.$$

Thus

$$\frac{1}{x}\int_{2}^{x} \frac{\mathrm{d}t}{\log t} = o(1) \quad (x \to \infty).$$

Using Theorem 2.4.3 we get

$$\begin{aligned}\frac{\vartheta(x)}{x} &= \frac{\pi(x)\log x}{x} - \frac{1}{x}\int_{2}^{x} \frac{\pi(t)}{t}\,\mathrm{d}t \\ &= \frac{\pi(x)\log x}{x} + \mathcal{O}\Big(\frac{1}{x}\int_{2}^{x} \frac{\mathrm{d}t}{\log t}\Big) \\ &= \frac{\pi(x)\log x}{x} + o(1).\end{aligned}$$

Therefore

$$\frac{\vartheta(x)}{x} \sim 1.$$

This means that (1) $\implies$ (2). Now suppose that $\vartheta(x) \sim x \quad (x \to \infty)$. Then,

$$\frac{\log x}{x}\int_{2}^{x} \frac{\vartheta(t)}{t\log^2 t}\,\mathrm{d}t = \mathcal{O}\Big(\frac{\log x}{x}\int_{2}^{x} \frac{\mathrm{d}t}{\log^2 t}\Big).$$

Now,

$$\int_{2}^{x} \frac{\mathrm{d}t}{\log^2 t} = \int_{2}^{\sqrt{x}} \frac{\mathrm{d}t}{\log^2 t} + \int_{\sqrt{x}}^{x} \frac{\mathrm{d}t}{\log^2 t} \leqslant \frac{\sqrt{x}-2}{\log^2 2} + \frac{x-\sqrt{x}}{\log^2\sqrt{x}}.$$

Thus

$$\frac{\log x}{x}\int_2^x \frac{\mathrm{d}t}{\log^2 t} = o(1) \quad (x \to \infty).$$

Using Theorem 2.4.3 we obtain

$$\begin{aligned}\frac{\pi(x)\log x}{x} &= \frac{\vartheta(x)}{x} + \frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t\log^2 t}\,\mathrm{d}t \\ &= \frac{\vartheta(x)}{x} + \mathcal{O}\Big(\frac{\log x}{x}\int_2^x \frac{\mathrm{d}t}{\log^2 t}\Big) \\ &= \frac{\vartheta(x)}{x} + o(1).\end{aligned}$$

This shows that

$$\frac{\pi(x)\log x}{x} \sim 1,$$

that is (2) $\implies$ (1). This completes the proof. $\square$

Chapter 3

The Riemann Zeta Function

This Chapter is based on [8] and [11]. The *Riemann Zeta function* is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathfrak{Re}(s) > 1). \tag{3.0.1}$$

By the integral test, the series converges absolutely and uniformly in any compact domain within $\mathfrak{R}(s) > 1$, hence ζ is holomorphic in this half-plane.

3.1 Analytic Continuation

The *Dirichlet eta function* is defined by the series

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \qquad (\mathfrak{Re}(s) > 0).$$

By the alternating series test, the series converges absolutely and uniformly in any compact domain within $\mathfrak{Re}(s) > 0$, hence η is holomorphic in this half-plane. Note that

$$\eta(s) = (1 - 2^{1-s})\zeta(s).$$

This yields analytic continuation of ζ to the half plane $\mathfrak{Re}(s) > 0$, except for a pole at $s = 1$. Note that if $\mathfrak{Re}(s) > 1$, then

$$\begin{aligned}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} &= \sum_{n=1}^{\infty} n\Big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big) \\ &= s\sum_{n=1}^{\infty} n\int_n^{n+1} \frac{\mathrm{d}x}{x^{s+1}}.\end{aligned}$$

Since $\lfloor x \rfloor = n$ for $x \in [n, n+1)$, we obtain

$$\zeta(s) = s\sum_{n=1}^{\infty}\int_n^{n+1} \frac{\lfloor x \rfloor}{x^{s+1}}\,\mathrm{d}x = s\int_1^{\infty} \frac{\lfloor x \rfloor}{x^{s+1}}\,\mathrm{d}x.$$

Let $\varphi\colon [1,\infty) \to [-\frac{1}{2}, \frac{1}{2}]$ be the periodic function defined by $\varphi(x) = x - \lfloor x \rfloor - \frac{1}{2}$. Then

$$\begin{aligned}\zeta(s) &= \frac{s}{s-1} - \frac{1}{2} - s\int_1^\infty \frac{\varphi(x)}{x^{s+1}}\,\mathrm{d}x \\ &= \frac{1}{s-1} + \frac{1}{2} - s(s+1)\int_1^\infty \left(\int_1^x \varphi(y)\,\mathrm{d}y\right)\frac{\mathrm{d}x}{x^{s+2}},\end{aligned} \tag{3.1.1}$$

where the last equality follows by partial integration. Since the integral of φ is bounded (it is an odd periodic function), then the last integral in x converges absolutely if $\mathfrak{Re}(s) > -1$, giving the analytic continuation of ζ to the half-plane $\mathfrak{Re}(s) > -1$. It also shows that $s = 1$ is a simple pole of ζ with residue 1, that is

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.$$

A graph of $|\zeta(s)|$ for $s \in \mathbb{C}$ is shown in Figure 3.1

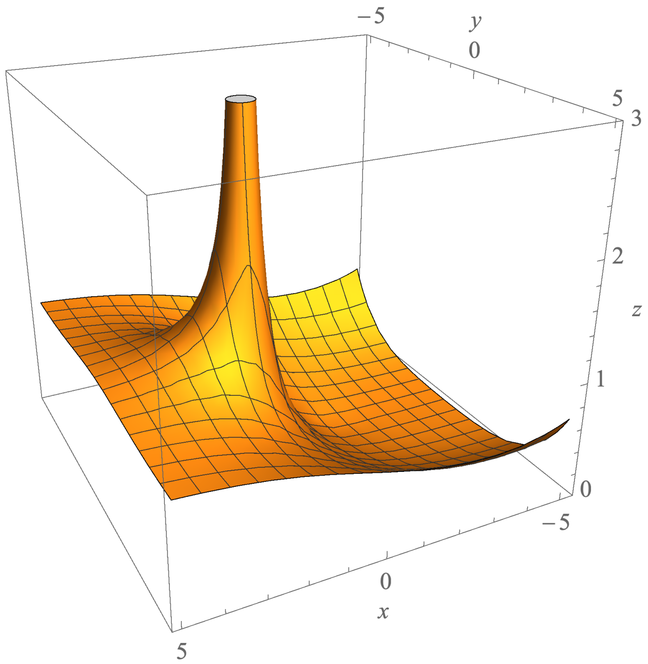

Figure 3.1: Absolute value of the Riemann zeta function

Note that

$$\zeta(0) = -\frac{1}{2}.$$

Moreover, using (3.1.1) we get

$$\lim_{s\to 1}\zeta(s) - \frac{1}{s-1} = \frac{1}{2} - \int_1^\infty \frac{\varphi(x)}{x^2}\,\mathrm{d}x = \frac{1}{2} - \lim_{N\to\infty}\int_1^N \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^2}\,\mathrm{d}x.$$

We evaluate the integral as follows:

$$\begin{aligned}\int_1^N \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^2}\,\mathrm{d}x &= \log N + \frac{1}{2N} - \frac{1}{2} - \sum_{n=1}^{N-1}\int_n^{n+1}\frac{n}{x^2}\,\mathrm{d}x \\ &= \log N + \frac{1}{2N} + \frac{1}{2} - \sum_{n=1}^{N}\frac{1}{n}.\end{aligned}$$

Therefore

$$\lim_{s\to 1} \zeta(s) - \frac{1}{s-1} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \log N = \gamma,$$

where γ is the Euler-Mascheroni constant. Hence

$$\zeta(s) = \frac{1}{s-1} + \gamma + \mathcal{O}(|s-1|) \qquad (s \to 1).$$

This shows that the function $\zeta(s) - \frac{1}{s-1} - \gamma$ has order $|s-1|$ near 1. For an illustration of this behaviour see the Figure below

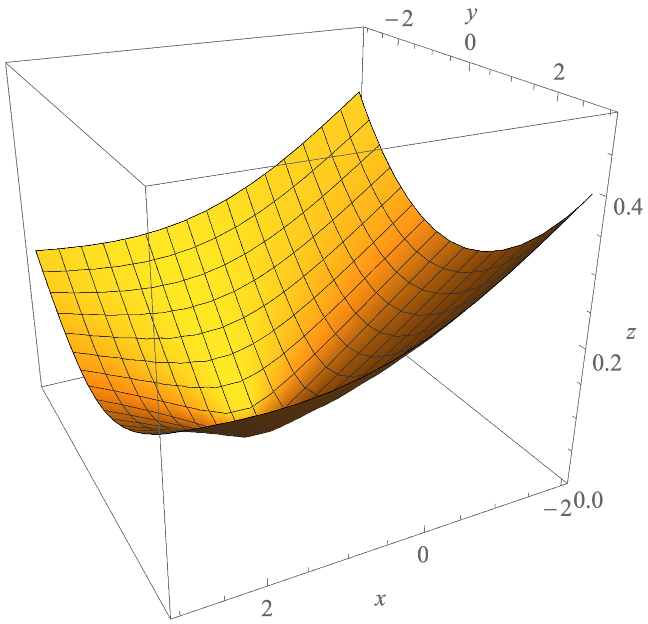

Figure 3.2: Absolute value of the function $\zeta(s) - \frac{1}{s-1} - \gamma$.

3.2 The Functional Equation

We follow the original ideas of Riemann. We know that

$$\Gamma\Big(\frac{s}{2}\Big) = \int_0^\infty \mathrm{e}^{-t}\, t^{\frac{s}{2}-1}\, \mathrm{d}t.$$

Letting $t = n^2\pi x$ we get

$$\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)n^{-s} = \int_0^\infty x^{\frac{s}{2}-1}\, \mathrm{e}^{-n^2\pi x}\, \mathrm{d}x.$$

Now we sum over all $n \in \mathbb{N}^*$ and use Theorem 1.1.1 to interchange summation and integration to obtain

$$\begin{aligned} \pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}\Big(\sum_{n=1}^{\infty} \mathrm{e}^{-n^2\pi x}\Big)\, \mathrm{d}x \\ &= \int_0^\infty x^{\frac{s}{2}-1}\Big(\frac{\theta(x)-1}{2}\Big)\, \mathrm{d}x, \end{aligned}$$

where

$$\theta(x) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{-n^2 \pi x}$$

is the Jacobi theta function. Thus

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \int_0^\infty x^{\frac{s}{2}-1} \Big(\frac{\theta(x) - 1}{2}\Big) \, \mathrm{d}x.$$

Now we write the integral in the form

$$\int_0^\infty x^{\frac{s}{2}-1} \Big(\frac{\theta(x)-1}{2}\Big) \, \mathrm{d}x = \int_0^1 x^{\frac{s}{2}-1} \Big(\frac{\theta(x)-1}{2}\Big) \, \mathrm{d}x + \int_1^\infty x^{\frac{s}{2}-1} \Big(\frac{\theta(x)-1}{2}\Big) \, \mathrm{d}x.$$

In the first integral, we let $u = \dfrac{1}{x}$ to get

$$\int_0^1 x^{\frac{s}{2}-1} \Big(\frac{\theta(x)-1}{2}\Big) \, \mathrm{d}x = \int_1^\infty u^{-\frac{s}{2}-1} \Big(\frac{\theta(u^{-1})-1}{2}\Big) \, \mathrm{d}u.$$

By Theorem 1.5.3 we know that

$$\theta(u^{-1}) = u^{\frac{1}{2}} \theta(u).$$

Thus

$$\begin{aligned}
\int_0^1 x^{\frac{s}{2}-1} \Big(\frac{\theta(x)-1}{2}\Big) \, \mathrm{d}x &= \frac{1}{2} \int_1^\infty u^{-\frac{s}{2}-\frac{1}{2}} \theta(u) \, \mathrm{d}u - \frac{1}{2} \int_1^\infty u^{-\frac{s}{2}-1} \, \mathrm{d}u \\
&= \frac{1}{2} \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}} \theta(x) \, \mathrm{d}x - \frac{1}{2} \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}} \, \mathrm{d}x + \frac{1}{s(s-1)}. \\
&= \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}} \Big(\frac{\theta(x)-1}{2}\Big) \, \mathrm{d}x + \frac{1}{s(s-1)}.
\end{aligned}$$

This shows that

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \left[\frac{1}{s(s-1)} + \int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}) \Big(\frac{\theta(x)-1}{2}\Big) \, \mathrm{d}x \right]. \tag{3.2.1}$$

Thanks to the exponential decay of θ, the above integral converges for all $s \in \mathbb{C}$ and hence defines an entire function. Since $\dfrac{1}{\Gamma}$ is an entire function and

$$\zeta(0) = -\frac{1}{2},$$

then (3.2.1) gives the analytic continuation of ζ to the whole complex plane with the exception of $s = 1$. Riemann noticed that formula (3.2.1) not only gives the analytic continuation of ζ, but can also be used to derive a functional equation for ζ. He observed that the term $\dfrac{1}{s(s-1)}$ and the integral in (3.2.1) are invariant under the transformation $s \to 1 - s$. Therefore

$$\pi^{-\frac{s}{2}} \Gamma\Big(\frac{s}{2}\Big) \zeta(s) = \pi^{\frac{1-s}{2}} \Gamma\Big(\frac{1-s}{2}\Big) \zeta(1-s). \tag{3.2.2}$$

Using the Legendre's duplication formula (2.1.7) and the Euler's reflection formula (2.1.5) it is easy to see that

$$\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} = \pi^{-\frac{1}{2}} 2^{1-s} \cos \frac{s\pi}{2} \Gamma(s),$$

and if this is used in the functional equation it gives the unsymmetrical form of the functional equation:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\Big(\frac{s\pi}{2}\Big) \Gamma(s) \zeta(s). \tag{3.2.3}$$

We can summarize the results in the following theorem:

Theorem 3.2.1 (The functional equation)**.** *The Riemann zeta function* $\zeta(s)$ *extends to a meromorphic function with a simple pole at* $s = 1$*, and the function*

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)$$

satisfies the functional equation

$$\xi(s) = \xi(1-s)$$

for every $s \in \mathbb{C}$*.*

The function ξ defined in Theorem 3.2.1 is called the *Riemann xi function* and is entire. As we know from Chapter 1, $\zeta(2m)$ can be expressed in terms of Bernoulli numbers (see (1.3.12)):

$$\zeta(2m) = \frac{(2\pi)^{2m}(-1)^{n+1}B_{2m}}{2(2m)!}, \qquad m = 1, 2, \ldots,$$

a relation known to Euler in 1737. Using the functional equation it is easy to see that in general we can write

$$\zeta(-m) = -\frac{B_{m+1}}{m+1}, \qquad m = 1, 2, \ldots.$$

3.3 The Euler Product Formula

To begin this section we give an infinite product representation for the Riemann zeta function, due to Euler. Suppose that $\mathfrak{Re}(s) > 1$. Subtracting the series $2^{-s}\zeta(s)$ from the one in (3.0.1) we obtain

$$(1 - 2^{-s})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots.$$

Similarly, we obtain

$$(1 - 2^{-s})(1 - 3^{-s})\zeta(s) = \sum \frac{1}{n^s},$$

where now the sum runs over $n \geqslant 1$, except for multiples of 2 and 3. Now, let p_m denote the m-th prime number. By repeating the above procedure we obtain

$$\zeta(s)\prod_{n=1}^{m}(1 - p_n^{-s}) = 1 + \sum \frac{1}{n^s},$$

where the sum runs over $n > 1$, except for the multiples of $p_1, p_2, \ldots, p_m$. Note that the sum of this series vanishes as $m \to \infty$ (since $p_m \to \infty$). Therefore, letting $m \to \infty$ we obtain *Euler's product formula*

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}} \qquad (\mathfrak{Re}(s) > 1). \tag{3.3.1}$$

This shows that ζ has no zeros in the half plane $\mathfrak{Re}(s) > 1$. Note that (3.2.1) shows that ζ has simple zeros at $s = -2, -4, \ldots$. These zeros, arising from the poles of the gamma function, are called the *trivial zeros*. From the functional equation (3.2.2) and using the fact that $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) > 1$, all other zeros, the *nontrivial zeros*, lie in the vertical strip $0 \leqslant \mathfrak{Re}(s) \leqslant 1$. Now, let $s \in \mathbb{C}$ with $s = \sigma + it$. Suppose that $\sigma > 1$. By taking logarithms of each side of (3.3.1) we obtain

$$\log \zeta(s) = -\sum_{p \in \mathbb{P}} \log(1 - p^{-s}).$$

Since

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad (|x|<1),$$

and $|p^{-s}| < 1$, then

$$\log \zeta(s) = \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} n^{-1} p^{-ns}. \tag{3.3.2}$$

Hence differentiating we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad (\mathfrak{Re}(s) > 1), \tag{3.3.3}$$

where Λ is the Mangoldt function defined by (3.7.1). Now we will show that ζ has no zeros in $\mathfrak{Re}(s) = 1$.

Theorem 3.3.1. *For every $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.*

Proof. Let $s \in \mathbb{C}$ with $s = \sigma + it$. Note that (3.3.2) can be written in the form

$$\log \zeta(s) = \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} n^{-1} p^{-n\sigma} \, \mathrm{e}^{-int\log p}.$$

Taking real part we get

$$\mathfrak{Re} \log \zeta(s) = \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} n^{-1} p^{-n\sigma} \cos(nt\log p). \tag{3.3.4}$$

Now, since $\mathfrak{Re}\log z = \log|z|$ for every $z \in \mathbb{C}$, using (3.3.4) we obtain

$$3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| = \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} n^{-1} p^{-n\sigma} \big(3 + 4\cos(nt\log p) + \cos(2nt\log p)\big).$$

Using the trigonometric identity

$$2(1+\cos\alpha)^2 = 3 + 4\cos\alpha + \cos(2\alpha) \geqslant 0,$$

valid for any $\alpha \in \mathbb{R}$, we get

$$3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \geqslant 0,$$

or equivalently

$$|\zeta(\sigma)|^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)| \geqslant 1.$$

Therefore

$$|(\sigma-1)\zeta(\sigma)|^3 \left| \frac{\zeta(\sigma+it)}{\sigma-1} \right|^4 |\zeta(\sigma+2it)| \geqslant \frac{1}{\sigma-1}. \tag{3.3.5}$$

Now we will show that $\zeta(1+it) \neq 0$ using inequality (3.3.5). Suppose by contradiction that $\zeta(1+it) = 0$ for some $t \in \mathbb{R}\setminus\{0\}$. Then, by L'Hôpital's rule we have

$$\lim_{\sigma\to 1} \frac{\zeta(\sigma+it)}{\sigma-1} = \zeta'(1+it).$$

Since ζ has a simple pole at $s = 1$, then

$$\lim_{\sigma\to 1}(\sigma - 1)\zeta(\sigma) = 1.$$

This shows that

$$\lim_{\sigma\to 1}|(\sigma-1)\zeta(\sigma)|^3\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4|\zeta(\sigma+2it)|$$

exists and equals $|\zeta'(1+it)|^4|\zeta(1+2it)|$. This contradicts (3.3.5). Hence we conclude that $\zeta(1+it) \neq 0$ for every $t \in \mathbb{R}$. $\square$

3.4 The Hadamard Product Formula over the Zeros

The goal of this section is to prove that the ξ function has the following product representation:

$$\xi(s) = \mathrm{e}^{A+Bs}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)\mathrm{e}^{\frac{s}{\rho}},$$

where $\rho = \beta + i\gamma$ runs over the nontrivial zeros of ζ. This result was conjectured by Riemann and proved by Hadamard in 1893.

Definition 3.4.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. We say that f has finite order if there exists positive constants α, β, λ such that*

$$|f(z)| \leqslant \alpha\,\mathrm{e}^{\beta|z|^\lambda} \tag{3.4.1}$$

for all $z \in \mathbb{C}$, or equivalently

$$|f(z)| \ll \exp\left(\beta|z|^\lambda\right) \qquad (z \to \infty).$$

The infimum of the exponents λ in (3.4.1) is called the order of f.

Lemma 3.4.1. *The Riemann xi function ξ defined by*

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

has order at most 1.

Proof. We first prove that

$$|\xi(s)| \ll \exp(C|s|\log|s|) \qquad (|s| \to \infty)$$

for some $C > 0$. Since

$$\xi(1-s) = \xi(s),$$

it suffices to prove the inequality for $\mathfrak{Re}(s) \geqslant \frac{1}{2}$. Clearly there exists some $\beta_1 > 0$ such that

$$\left|\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\right| \ll \exp(\beta_1|s|) \qquad (|s| \to \infty).$$

Since $|\arg z| < \pi$, we may apply Stirling's formula (2.2.13) to get

$$\left|\Gamma\left(\frac{s}{2}\right)\right| \ll \exp(\beta_2|s|\log|s|) \qquad (|s| \to \infty),$$

for some for some $\beta_2 > 0$. Using the integral representation for ζ obtained in (3.1.1) we get

$$|\zeta(s)| \ll |s| = \exp(\log |s|).$$

This shows that there exists $C > 0$ such that

$$|\xi(s)| = \left|\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\right| \ll \exp(C|s|\log|s|) \qquad (|s| \to \infty).$$

Now, we recall from Example 1.4.1 that for all $\varepsilon > 0$ we have

$$\log|s| \ll |s|^{\varepsilon} \qquad (s \to \infty).$$

This shows that there is some $\beta > 0$ such that for all $\varepsilon > 0$, we have

$$|\xi(s)| \ll \exp\left(\beta|s|^{1+\varepsilon}\right) \qquad (|s| \to \infty).$$

Therefore ξ has order at most 1. □

Lemma 3.4.2. *An entire function $f : \mathbb{C} \to \mathbb{C}$ of order κ which does not have zeros has the form*

$$f(z) = \mathrm{e}^{g(z)},$$

where g is a polynomial of degree κ.

Proof. Since f is entire and $f(z) \neq 0$ for all $z \in \mathbb{C}$, then the function $\log f(z)$ exists and is entire, so

$$g(z) = \log f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where the power series converges absolutely. This shows that

$$f(z) = \mathrm{e}^{g(z)}.$$

Now we show that g is a polynomial of degree κ. Since f has finite order, then using (3.4.1), we get

$$\mathfrak{Re}\, g(z) = \log|f(z)| \leqslant \beta|z|^{\lambda} + \log\alpha. \tag{3.4.2}$$

Let $a_m = b_m + ic_m$. In the circle $|z| = R\,\mathrm{e}^{2\pi i\theta}$, $0 \leqslant \theta < 1$, we have

$$\mathfrak{Re}\, g(z) = b_0 + \sum_{m=1}^{\infty}(b_m \cos 2\pi m\theta - c_m \sin 2\pi m\theta)R^m.$$

Using the orthogonality relations of the trigonometric functions (1.2.3) and (1.2.4) we get

$$b_n R^n = 2\int_0^1 \mathfrak{Re}\, g(R\,\mathrm{e}^{2\pi i\theta}) \cos 2\pi n\theta \,\mathrm{d}\theta.$$

Therefore, using (3.4.2) we obtain

$$\begin{aligned}
|b_n|R^n &\leqslant 2\int_0^1 |\mathfrak{Re}\, g(R\,\mathrm{e}^{2\pi i\theta})| \,\mathrm{d}\theta \\
&= 2\int_0^1 \left[|\mathfrak{Re}\, g(R\,\mathrm{e}^{2\pi i\theta})| + \mathfrak{Re}\, g(R\,\mathrm{e}^{2\pi i\theta})\right] \mathrm{d}\theta - 2b_o \\
&= 4\int_0^1 \max\{0, \mathfrak{Re}\, g(R\,\mathrm{e}^{2\pi i\theta})\} \,\mathrm{d}\theta - 2b_0 \\
&\leqslant 4(\beta R^{\lambda} + \log\alpha) - 2b_0.
\end{aligned}$$

Since R can be arbitrarily large, then $b_n = 0$ for $n > \lambda$. Similarly $c_n = 0$ for $n > \lambda$. This shows that g is a polynomial of degree less than or equal to λ. Since κ is the infimum of the exponents λ, then $\kappa = \deg g$. □

The next result gives a connection between the zeros of an analytic function f and $\log|f|$.

Lemma 3.4.3 (Jensen's Formula)**.** *Let $f\colon \mathbb{C} \to \mathbb{C}$ be analytic on the disk $|z| \leqslant R$. If $f(0) \neq 0$ and $f(z) \neq 0$ for $|z| = R$, then*

$$\int_0^1 \log|f(R\,\mathrm{e}^{2\pi i\theta})|\,\mathrm{d}\theta = \log\frac{|f(0)|R^n}{|z_1 z_2 \cdots z_n|} = \int_0^R \frac{n(r)}{r}\,\mathrm{d}r + \log|f(0)|,$$

where $z_1, z_2, \ldots, z_n$ are all the zeros of f (counted with multiplicities) and $n(r)$ is the number of zeros in $|z| \leqslant r$.

Proof. If g is analytic in $|z| \leqslant R$ and $g(z) \neq 0$ then, by Cauchy's integral theorem we have

$$\log g(0) = \frac{1}{2\pi i}\int_{|z|=R} \frac{\log g(z)}{z}\,\mathrm{d}z.$$

Therefore, taking the real part we get

$$\log|g(0)| = \int_0^1 \log\left|g(R\,\mathrm{e}^{2\pi i\theta})\right|\mathrm{d}\theta.$$

Let

$$g(z) \coloneqq f(z)\prod_{k=1}^{n}\frac{R^2 - z\,\overline{z_k}}{R(z - z_k)}.$$

Note that if $|z| = R$, then

$$\left|\frac{R^2 - z\,\overline{z_k}}{R(z-z_k)}\right| = \left|\frac{z(\overline{z} - \overline{z_k})}{R(z-z_k)}\right| = 1.$$

Thus $|g(z)| = |f(z)|$ for $|z| = R$. Note that g is analytic in $|z| \leqslant R$ and $g(z) \neq 0$. This shows that

$$\begin{aligned}\int_0^1 \log|f(R\,\mathrm{e}^{2\pi i\theta})|\,\mathrm{d}\theta &= \int_0^1 \log|g(R\,\mathrm{e}^{2\pi i\theta})|\,\mathrm{d}\theta \\ &= \log|g(0)| \\ &= \log\frac{|f(0)|R^n}{|z_1 z_2 \cdots z_n|}.\end{aligned}$$

Now suppose without loss of generality that $|z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_n| \leqslant |z_{n+1}| \coloneqq R$. Then

$$\begin{aligned}\int_0^R \frac{n(r)}{r}\,\mathrm{d}r &= \sum_{k=1}^{n}\int_{|z_k|}^{|z_{k+1}|}\frac{k}{r}\,\mathrm{d}r \\ &= \sum_{k=1}^{n} k(\log|z_{k+1}| - \log|z_k|) \\ &= \log\frac{R^n}{|z_1 z_2 \cdots z_n|}.\end{aligned}$$

□

Corollary 3.4.0.1. *If $f\colon \mathbb{C} \to \mathbb{C}$ is entire and satisfies (3.4.1), then the number of zeros in the disk $|z| \leqslant R$ satisfies*

$$n(R) \ll R^\lambda \qquad (R \to \infty).$$

Proof. Let κ be the order of f. If $\lambda > \kappa$, then using (3.4.1) we get

$$\log |f(Re^{2\pi i\theta})| < R^{\gamma},$$

for R sufficiently large. Now, by Jensen's formula (3.4.3) we have

$$\int_0^R \frac{n(r)}{r}\,\mathrm{d}r < R^{\lambda} - \log|f(0)| < 2R^{\lambda}.$$

Since

$$\int_R^{2R} \frac{n(r)}{r}\,\mathrm{d}r \geqslant n(R)\int_R^{2R}\frac{\mathrm{d}r}{r} = n(R)\log 2,$$

it follows that

$$n(R) \leqslant \frac{1}{\log 2}\int_0^R \frac{n(r)}{r}\,\mathrm{d}r < \frac{2}{\log 2}R^{\lambda}.$$

This shows that $n(R) = \mathcal{O}(R^{\lambda})$ as $R \to \infty$. □

Corollary 3.4.0.2. *If $f\colon \mathbb{C} \to \mathbb{C}$ is entire and satisfies (3.4.1), then for every $\varepsilon > 0$,*

$$\sum_{\rho\neq 0} |\rho|^{-\lambda-\varepsilon} < \infty,$$

where ρ runs over the zeros of f.

Proof. Let $\nu > \lambda$. Let $z_1, z_2 \ldots$ be the zeros of f (finite or countable zeros). Let $r_k := |z_k|$ and $r_0 := 0$. Then

$$\sum_{\rho\neq 0} |\rho|^{-\nu} = \sum_{j=1}^{\infty}\big(n(r_j) - n(r_{j-1})\big)r_j^{-\nu}.$$

By expanding the sum it is easy to see that

$$\sum_{j=1}^{\infty}\big(n(r_j) - n(r_{j-1})\big)r_j^{-\nu} = \sum_{j=1}^{\infty} n(r_j)\big(r_j^{-\nu} - r_{j+1}^{-\nu}\big).$$

Therefore

$$\sum_{\rho\neq 0} |\rho|^{-\nu} = \sum_{j=1}^{\infty} n(r_j)\int_{r_j}^{r_{j+1}} \nu r^{-\nu-1}\,\mathrm{d}r = \nu\int_0^{\infty} n(r)r^{-\nu-1}\,\mathrm{d}r.$$

By Corollary (3.4.0.1) we conclude that

$$\sum_{\rho\neq 0} |\rho|^{-\nu} \ll \int_0^{\infty} r^{\lambda-\nu-1}\,\mathrm{d}r < \infty.$$

The particular case $\nu = \lambda + \varepsilon$ completes the proof. □

From now on we restrict our analysis to entire functions f with order at most one, since this is the only case which we shall be concerned later. Since the series

$$\sum_{\rho\neq 0} |\rho|^{-1-\varepsilon}$$

converges, then

$$\sum_{\rho\neq 0} \log\left(1 - \frac{z}{\rho}\right) + \frac{z}{\rho} \ll \sum_{\rho\neq 0} |\rho|^{-2} < \infty.$$

Therefore we can define the product

$$P(z) = \prod_{\rho \neq 0} \Big(1 - \frac{z}{\rho}\Big)\, \mathrm{e}^{\frac{z}{\rho}}\,. \tag{3.4.3}$$

Since the total length of all the intervals $(|\rho| - |\rho|^{-2}, |\rho| + |\rho|^{-2})$ on the real line is finite, there are arbitrarily large numbers R such that

$$\big|R - |\rho|\big| > |\rho|^{-2} \qquad \forall \rho \neq 0. \tag{3.4.4}$$

Let $P(z) = P_1(z)P_2(z)P_3(z)$, where these are the subproducts extended over the following sets:

$$\begin{aligned} P_1 :& \qquad |\rho| < \frac{1}{2}R, \\ P_2 :& \qquad \frac{1}{2}R \leqslant |\rho| \leqslant 2R, \\ P_3 :& \qquad |\rho| > 2R. \end{aligned}$$

For the factors of P_1, on $|z| = R$, we have

$$\left|\Big(1 - \frac{z}{\rho}\Big)\, \mathrm{e}^{\frac{z}{\rho}}\,\right| \geqslant \left(\Big|\frac{z}{\rho}\Big| - 1\right) \mathrm{e}^{-\frac{|z|}{|\rho|}} > \mathrm{e}^{-\frac{R}{|\rho|}},$$

and since

$$\sum_{|\rho| < \frac{1}{2}R} |\rho|^{-1} < \Big(\frac{1}{2}R\Big)^{\varepsilon} \sum_{|\rho| \neq 0} |\rho|^{-1-\varepsilon},$$

it follows that

$$\log|P_1(z)| > -R \sum_{0 < |\rho| < \frac{1}{2}R} |\rho|^{-1} \gg -R^{1+\varepsilon}.$$

Thus

$$|P_1(z)| \gg \exp(-R^{1+\varepsilon}).$$

For the factors of P_2, using (3.4.4) we get

$$\left|\Big(1 - \frac{z}{\rho}\Big)\, \mathrm{e}^{\frac{z}{\rho}}\,\right| \geqslant \mathrm{e}^{-2}\, \frac{|z - \rho|}{2R} \gg R^{-3}.$$

By Corollary 3.4.0.1 the numbers of factors is less than $R^{1+\frac{\varepsilon}{2}}$. Hence

$$|P_2(z)| \gg (R^{-3})^{R^{1+\frac{\varepsilon}{2}}} \gg \exp(-R^{1+\varepsilon}).$$

Finally, for the factors of P_3, note that there exists some $C > 0$ such that

$$\log\left|\Big(1 - \frac{z}{\rho}\Big)\, \mathrm{e}^{\frac{z}{\rho}}\,\right| \geqslant -\left|\log\Big(1 - \frac{z}{\rho}\Big) + \frac{z}{\rho}\right| \geqslant -C\Big(\frac{R}{|\rho|}\Big)^2.$$

Hence

$$\left|\Big(1 - \frac{z}{\rho}\Big)\, \mathrm{e}^{\frac{z}{\rho}}\,\right| \geqslant \mathrm{e}^{-C(\frac{R}{|\rho|})^2}\,.$$

We also have

$$\sum_{|\rho| > 2R} |\rho|^{-2} < (2R)^{-1+\varepsilon} \sum_{\rho \neq 0} |\rho|^{-1-\varepsilon},$$

and therefore

$$|P_3(z)| \gg \exp(-R^{1+\varepsilon}).$$

It follows that , on $|z| = R$, we have

$$|P(z)| \gg \exp(-R^{1+\varepsilon}).$$

Let $F : \mathbb{C} \to \mathbb{C}$ be defined as $F(z) = \frac{f(z)}{P(z)}$. Then, F is an entire function which has no zeros and satisfies

$$|F(z)| \ll \exp(R^{1+\varepsilon}), \qquad \text{on } |z| = R.$$

Since R can be arbitrarily large, by the maximum principle this estimate implies that F is of order at most one. So by Lemma 3.4.2 we conclude that $F(z) = \mathrm{e}^{g(z)}$, where g is a polynomial of degree at most one. We can summarize the results in the following theorem:

Theorem 3.4.1. *An entire function f of order at most one with $f(0) \neq 0$ has the following product representation:*

$$f(z) = \mathrm{e}^{A+Bz} \prod_{\rho} \left(1 - \frac{z}{\rho}\right) \mathrm{e}^{\frac{z}{\rho}}.$$

Corollary 3.4.1.1. *For every $z \in \mathbb{C}$ we have*

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right).$$

Proof. For every $z \in \mathbb{C}$ let

$$f(z) := \frac{\sin z}{z}.$$

Then

$$|f(z)| = \left|\frac{e^{iz} - e^{-iz}}{2iz}\right| \ll \exp\left(\beta |z|^{1+\varepsilon}\right).$$

This shows that f has order at most one. Since $f(0) \neq 0$ and the roots of $\sin z$ are integer multiples of π then Theorem 3.4.1 shows that f has the following product representation

$$f(z) = \mathrm{e}^{A+Bz} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right). \tag{3.4.5}$$

Since $f(0) = 1$ we get $A = 0$. Taking the logarithmic derivative of (3.4.5) we get

$$\frac{z\cos z - \sin z}{z \sin z} = B + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2\pi^2}.$$

Letting $z \to 0$ it is easy to see that $B = 0$. This completes the proof. □

Now, since ξ has order at most one (Lemma 3.4.1) we conclude that

$$\xi(s) = \mathrm{e}^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \mathrm{e}^{\frac{s}{\rho}}.$$

Taking the logarithmic derivative, we get

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Since

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s),$$

then

$$\frac{\zeta'(s)}{\zeta(s)} = -B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\psi\Big(\frac{s}{2}+1\Big) + \sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big),$$

where ψ is the digamma function. For $s = \sigma + it$ with $-1 \leqslant \sigma \leqslant 2$, $t \geqslant 2$, by Stirling's series (2.2.12) we get

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) + \mathcal{O}(\log t). \tag{3.4.6}$$

By (3.3.3), $\dfrac{\zeta'(s)}{\zeta(s)}$ is bounded for $s = 2 + iT$, $T \geqslant 2$. Hence for such s we have

$$\sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) \ll \log T. \tag{3.4.7}$$

If $\rho = \beta + i\gamma$[1] with $0 \leqslant \beta \leqslant 1$, then taking the real part of (3.4.7) we get

$$\sum_\rho \frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2} + \frac{\beta}{\beta^2+\gamma^2} \ll \log T.$$

Since

$$\frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2} \geqslant \frac{1}{4+(T-\gamma)^2} \gg \frac{1}{1+(T-\gamma)^2},$$

and $\beta \geqslant 0$, then

$$\sum_\rho \frac{1}{1+(T-\gamma)^2} \ll \log T.$$

Therefore, the number of zeros $\rho = \beta + i\gamma$ of ζ in the box $0 < \beta < 1$, $T-1 < \gamma \leqslant T+1$ satisfies

$$\begin{aligned} N(T+1) - N(T-1) &\leqslant 2\sum_{T-1<\gamma\leqslant T+1}\frac{1}{1+(T-\gamma)^2} \\ &\leqslant 2\sum_\rho \frac{1}{1+(T-\gamma)^2} \\ &\ll \log T. \end{aligned} \tag{3.4.8}$$

For future references we write the previous result in the following Lemma:

Lemma 3.4.4. *The number of zeros $\rho = \beta + i\gamma$ of ζ in the box $0 < \beta < 1$, $T < \gamma \leqslant T+1$ satisfies*

$$N(T+1) - N(T) \ll \log T$$

Proof. We have

$$N(T+1) - N(T) \leqslant N(T+1) - N(T-1) \ll \log T.$$ □

Theorem 3.4.2. *For $s = \sigma + it$ in the strip $-1 \leqslant \sigma \leqslant 2$, we have*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|<1}\frac{1}{s-\rho} + \mathcal{O}(\log t) \qquad (t\to\infty),$$

where the sum runs over the nontrivial zeros $\rho = \beta + i\gamma$ of ζ for which $|t-\gamma| < 1$.

[1]Do not confuse with the Euler-Mascheroni constant which is also denoted by γ.

Proof. Using (3.4.6) we obtain

$$\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(2+it)}{\zeta(2+it)} = \sum_{\rho}\Big(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\Big) + \mathcal{O}(\log t).$$

Since $\frac{\zeta'(2+it)}{\zeta(2+it)}$ is bounded for large t by (3.3.3), then

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho}\Big(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\Big) + \mathcal{O}(\log t).$$

For the terms with $|t-\gamma| \geqslant 1$, we have

$$\Big|\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\Big| = \frac{2-\sigma}{|(s-\rho)(2+it-\rho)|} \leqslant \frac{3}{(t-\gamma)^2},$$

and therefore,

$$\Big|\sum_{|t-\gamma|\geqslant 1}\Big(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\Big)\Big| \leqslant \sum_{|t-\gamma|\geqslant 1}\frac{3}{(t-\gamma)^2} \ll \sum_{\rho}\frac{1}{1+(t-\gamma)^2} \ll \log t.$$

For the terms with $|t-\gamma| < 1$, we have $|2+it-\rho| \geqslant 1$. Thus, using (3.4.8) we obtain

$$\Big|\sum_{|t-\gamma|<1}\frac{1}{2+it-\rho}\Big| \leqslant \sum_{t-1<\gamma<t+1} 1 \ll N(t+1) - N(t-1) \ll \log t.$$

This completes the proof. □

3.5 The Asymptotic Formula for $N(T)$

The purpose of this section is to count the number of zeros of ζ in the critical strip with imaginary part $|\mathfrak{Im}(s)| < T$ for any positive real number T. Once we know how many zeros should lie in a region, we can verify the Riemann hypothesis in that region computationally. For this purpose we need the argument principle, for a proof see [2].

Theorem 3.5.1 (The argument principle)**.** *Let f be meromorphic in a domain interior to a positively oriented simple contour $\mathcal{L}$ such that f is analytic and nonzero on $\mathcal{L}$. Then*

$$\Delta_{\mathcal{L}} \arg f(z) = 2\pi(Z-P),$$

where Z and P are the number of zeros and the number of poles of f inside $\mathcal{L}$, counting multiplicities, and $\Delta_{\mathcal{L}} \arg f(z)$ counts the changes in the argument of f along the contour $\mathcal{L}$.

Using the argument principle we are ready to prove the following theorem which was conjectured by Riemann and proved by von Mangoldt.

Theorem 3.5.2. *Let $N(T)$ be the number of zeros $\rho = \beta + i\gamma$ of ζ in the rectangle*

$$0 < \beta < 1, \quad 0 < \gamma \leqslant T.$$

Then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T) \qquad (T \to \infty). \tag{3.5.1}$$

Proof. Since ζ and ξ have the same zeros (counted with multiplicity), we shall work with the entire function ξ rather than with ζ. Let $\mathcal{L}$ be the positively oriented rectangle with vertices $-1, 2, 2+iT, -1+iT$, as in Figure 3.3.

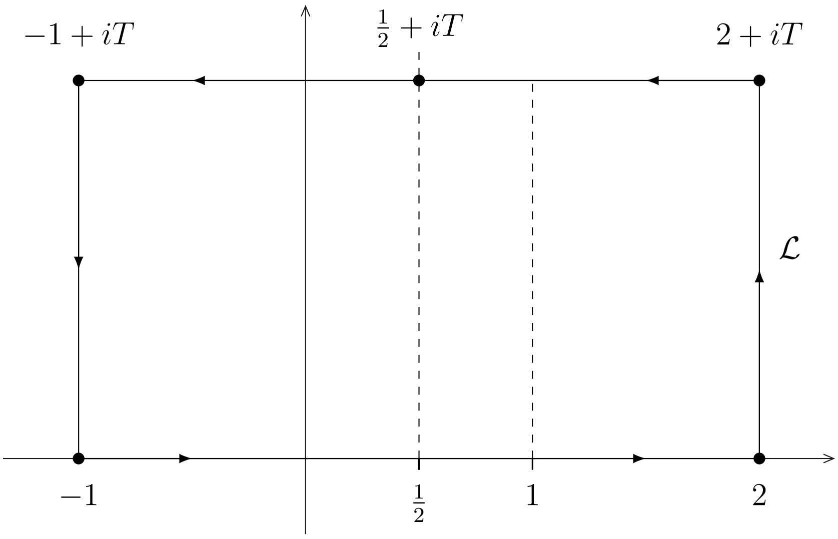

Figure 3.3: Contour of integration $\mathcal{L}$

By the argument principle (Theorem 3.5.1) we have

$$\Delta_{\mathcal{L}} \arg \xi(s) = 2\pi N(T).$$

Let $\mathcal{M}$ be the part of $\mathcal{L}$ to the right of the critical line $\mathfrak{Re}\, s = \dfrac{1}{2}$. By the functional equation $\xi(s) = \xi(1-s)$ it follows that

$$\Delta_{\mathcal{L}} \arg \xi(s) = 2\Delta_{\mathcal{M}} \arg \xi(s).$$

Hence

$$N(T) = \frac{1}{\pi}\Delta_{\mathcal{M}} \arg \xi(s).$$

Now we compute the variation of the argument of

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)$$

for each factor separately along every segment. There is no variation from the line segment from $\dfrac{1}{2}$ to 1, because every factor is real. Now,

$$\Delta_{\mathcal{M}} \arg \frac{1}{2}s(s-1) = \arg\Big(\frac{1}{2}+iT\Big)\Big(-\frac{1}{2}+iT\Big) = \arg -\Big(\frac{1}{4}+T^2\Big) = \pi,$$

and

$$\Delta_{\mathcal{M}} \arg \pi^{-\frac{s}{2}} = \arg \pi^{-\frac{1}{4}-i\frac{T}{2}} = -\frac{T}{2}\log \pi.$$

To compute the argument variation of Γ we use Stirling's series (2.2.12) in the form

$$\log\Gamma(s) = \Big(s-\frac{1}{2}\Big)\log s - s + \frac{1}{2}\log 2\pi + \mathcal{O}\Big(\frac{1}{|s|}\Big),$$

to get

$$\begin{aligned}\Delta_{\mathcal{M}} \arg \Gamma\Big(\frac{s}{2}\Big) &= \mathfrak{Im} \log \Gamma\Big(\frac{1}{4} + i\frac{T}{2}\Big) \\ &= \mathfrak{Im}\Big\{\Big(-\frac{1}{4} + i\frac{T}{2}\Big) \log\Big(-\frac{1}{4} + i\frac{T}{2}\Big) - \Big(-\frac{1}{4} + i\frac{T}{2}\Big) + \frac{1}{2}\log 2\pi + \mathcal{O}\Big(\frac{1}{T}\Big)\Big\} \\ &= -\frac{\pi}{8} + \frac{T}{2}\log\frac{T}{2} - \frac{T}{2} + \mathcal{O}\Big(\frac{1}{T}\Big).\end{aligned}$$

Adding up the above results we obtain

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + \mathcal{O}\Big(\frac{1}{T}\Big), \tag{3.5.2}$$

where

$$S(T) := \frac{1}{\pi}\Delta_{\mathcal{M}} \arg\zeta(s) = \frac{1}{\pi}\arg\zeta\Big(\frac{1}{2} + iT\Big),$$

provided this argument is defined by continuous variation along $\mathcal{L}$, or, equivalently, by continuous variation along the horizontal segment from $\infty + iT$ to $\frac{1}{2} + iT$ starting at $\infty + iT$ with the value 0. The formula (3.5.1) will follow from (3.5.2) if we show that

$$S(T) \ll \log T.$$

To estimate $S(T)$ we use the integral

$$\pi S(T) = \int_{\infty+iT}^{\frac{1}{2}+iT} \mathfrak{Im}\frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s. \tag{3.5.3}$$

For $s = \sigma + iT$ with $2 \leqslant \sigma < \infty$ we use (3.3.3) to get

$$\Big|\frac{\zeta'(s)}{\zeta(s)}\Big| \leqslant \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}.$$

Therefore, using Theorem 1.1.1 to interchange summation and integration we conclude that the integral (3.5.3) along the segment from $\infty + iT$ to $2 + iT$ is bounded by the constant

$$\int_2^{\infty}\Big(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}\Big)\,\mathrm{d}\sigma = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2\log n}.$$

Now,

$$\Big|\int_{2+iT}^{\frac{1}{2}+iT}\mathfrak{Im}\Big(\frac{1}{s-\rho}\Big)\,\mathrm{d}s\Big| = |\Delta\arg(s-\rho)| \leqslant \pi.$$

Therefore, using Theorem 3.4.2 we get

$$\int_{2+iT}^{\frac{1}{2}+iT}\mathfrak{Im}\frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s \ll \sum_{|t-\gamma|<1}\int_{2+iT}^{\frac{1}{2}+iT}\mathfrak{Im}\Big(\frac{1}{s-\rho}\Big)\,\mathrm{d}s + \log T \ll \sum_{|\gamma-T|<1} 1 + \log T.$$

By (3.4.8) the number of zeros with $|\gamma - T| < 1$ is $\mathcal{O}(\log T)$. Thus

$$S(T) \ll \log T,$$

and finally we conclude that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T).$$

This completes the proof. □

3.6 The Hardy's Theorem

In 1914, Hardy proved that *there are infinitely many roots s of $\zeta(s)$ with real part* $\mathfrak{Re}(s) = \dfrac{1}{2}$. This theorem provides one of the best results towards the validity of the Riemann hypothesis because it establishes the most basic necessary condition for the Riemann hypothesis to be true.

Theorem 3.6.1. *There are infinitely many zeros of ζ on the critical line.*

Proof. Let $\Xi : \mathbb{C} \to \mathbb{C}$ be the function defined as $\Xi(z) = \xi\Big(\dfrac{1}{2} + iz\Big)$. If $t \in \mathbb{R}$, then by the functional equation we have

$$\Xi(t) = \xi\Big(\frac{1}{2} + it\Big) = \xi\Big(\frac{1}{2} - it\Big) = \overline{\xi\Big(\frac{1}{2} + it\Big)}.$$

This shows that $\Xi(t) \in \mathbb{R}$ for $t \in \mathbb{R}$. Therefore, a zero of $\zeta(s)$ on $\mathfrak{Re}(s) = \dfrac{1}{2}$ corresponds to a real zero of $\Xi(t)$. To prove the theorem we will show that Ξ has infinite real zeros. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(t) = |g(it)|^2 = g(it)g(-it)$, where g is analytic. Consider the integral

$$\Phi(x) = \int_0^\infty f(t)\Xi(t)\cos xt \, \mathrm{d}t.$$

Letting $y = \mathrm{e}^x$, $s = \dfrac{1}{2} + it$ and using the fact that Ξ is an even function we get

$$\begin{aligned}
\Phi(x) &= \frac{1}{2}\int_{-\infty}^{\infty} g(it)g(-it)\Xi(t)y^{it}\, \mathrm{d}t \\
&= \frac{1}{2\sqrt{y}i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} g\Big(s-\frac{1}{2}\Big)g\Big(\frac{1}{2}-s\Big)\xi(s)y^s \, \mathrm{d}s \\
&= \frac{1}{2\sqrt{y}i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} g\Big(s-\frac{1}{2}\Big)g\Big(\frac{1}{2}-s\Big)(s-1)\Gamma\Big(1+\frac{s}{2}\Big)\pi^{-\frac{s}{2}}\zeta(s)y^s \, \mathrm{d}s.
\end{aligned}$$

Taking $g(s) = \dfrac{1}{s + \frac{1}{2}}$ we obtain

$$\begin{aligned}
\Phi(x) &= -\frac{1}{2\sqrt{y}i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{s}\Gamma\Big(1+\frac{s}{2}\Big)\pi^{-\frac{s}{2}}\zeta(s)y^s \, \mathrm{d}s \\
&= -\frac{1}{4\sqrt{y}i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\Big(\frac{s}{2}\Big)\pi^{-\frac{s}{2}}\zeta(s)y^s \, \mathrm{d}s.
\end{aligned}$$

Consider the positively oriented rectangle with vertices at $\dfrac{1}{2} - iR, \dfrac{1}{2} + iR, 2 - iR, 2 - iR$, as shown the Figure 3.4.

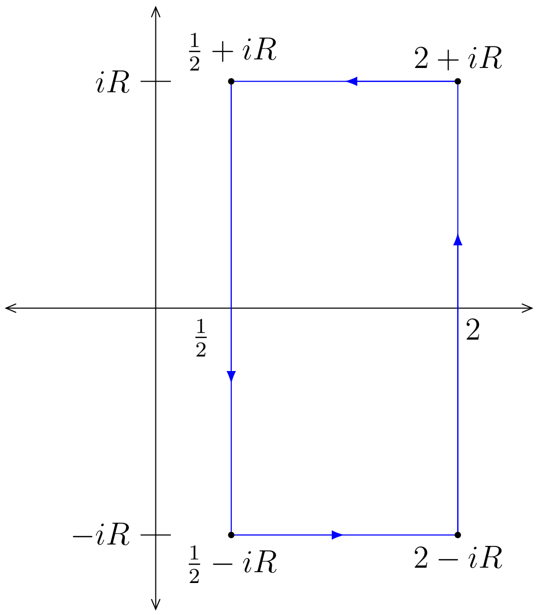

Figure 3.4: Rectangular contour

Since

$$2\pi i \operatorname*{Res}_{s=1}\left\{\frac{1}{4\sqrt{y}i}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)y^s\right\} = \frac{\pi}{2}\sqrt{y},$$

then letting $R \to \infty$, by the Residue Theorem it follows that

$$\begin{aligned}\Phi(x) &= -\frac{1}{4\sqrt{y}i}\int_{2-i\infty}^{2+i\infty}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)y^s\,\mathrm{d}s + \frac{\pi}{2}\sqrt{y}\\ &= -\frac{1}{4\sqrt{y}i}\sum_{n=1}^{\infty}\int_{2-i\infty}^{2+i\infty}\Gamma\left(\frac{s}{2}\right)\left(\frac{\sqrt{\pi}n}{y}\right)^{-s}\mathrm{d}s + \frac{\pi}{2}\sqrt{y},\end{aligned}$$

where the last equality follows by the definition of $\zeta(s)$ for $\mathfrak{Re}(s) > 1$, and we used Theorem 1.1.1 to interchange summation and integration. Now, by Mellin inversion (Theorem 1.5.5) we know that for any c is

$$\mathrm{e}^{-t} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}\Gamma(s)\,\mathrm{d}s.$$

Therefore

$$\Phi(x) = -\frac{\pi}{\sqrt{y}}\varphi\left(\frac{1}{y^2}\right) + \frac{\pi}{2}\sqrt{y},$$

where

$$\varphi(t) = \sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi t}.$$

This shows that

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}\cos xt\,\mathrm{d}t = \frac{\pi}{2}\left\{\mathrm{e}^{\frac{x}{2}} - 2\,\mathrm{e}^{-\frac{x}{2}}\varphi(\mathrm{e}^{-2x})\right\}.$$

Letting $x = i\alpha$ we get

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}\cosh \alpha t\,\mathrm{d}t = 2\cos\frac{\alpha}{2} - 2\,\mathrm{e}^{\frac{i\alpha}{2}}\left\{\frac{1}{2} + \varphi(\mathrm{e}^{2i\alpha})\right\}. \tag{3.6.1}$$

Note that if $z = x + iy$, $x \geqslant 0$, then

$$\begin{aligned}
|z^{z-\frac{1}{2}}\,\mathrm{e}^{-z}| &= |z|^{x-\frac{1}{2}}\,\mathrm{e}^{-x-|y|\arg z}\\
&= |z|^{x-\frac{1}{2}}\,\mathrm{e}^{-\frac{\pi}{2}|y|}\exp\Big\{-x-|y|\Big(\arg z-\frac{\pi}{2}\Big)\Big\}\\
&\leqslant |z|^{x-\frac{1}{2}}\,\mathrm{e}^{-\frac{\pi}{2}|y|}.
\end{aligned}$$

Hence, by Stirling's formula (2.2.13) we have

$$\Gamma\Big(\frac{1}{4}+\frac{1}{2}it\Big) \ll \Big(\frac{1}{4}+\frac{1}{2}it\Big)^{-\frac{1}{2}+\frac{1}{2}it}\mathrm{e}^{-\frac{1}{4}-\frac{1}{2}it} \ll t^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{\pi t}{4}}. \tag{3.6.2}$$

Now, since

$$\Xi(t) = -\frac{1}{2}\Big(t^2+\frac{1}{4}\Big)\pi^{-\frac{1}{4}-\frac{1}{2}it}\Gamma\Big(\frac{1}{4}+\frac{1}{2}it\Big)\zeta\Big(\frac{1}{2}+it\Big),$$

then using the integral representation for ζ (3.1.1) and (3.6.2) we get

$$\Xi(t) \ll t^{\beta}\,\mathrm{e}^{-\frac{\pi t}{4}},$$

for some $\beta > 0$. This shows that the integral in (3.6.1) may be differentiated with respect to α any number of times provided that $\alpha < \frac{\pi}{4}$. Thus

$$\frac{2}{\pi}\int_0^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,\mathrm{d}t = \frac{(-1)^n\cos\frac{\alpha}{2}}{2^{2n-1}} - 2\frac{\mathrm{d}^{2n}}{\mathrm{d}\alpha^{2n}}\,\mathrm{e}^{\frac{i\alpha}{2}}\Big\{\frac{1}{2}+\varphi(\mathrm{e}^{2i\alpha})\Big\}. \tag{3.6.3}$$

Now we shall prove that for all $n \in \mathbb{N}$, we have

$$\frac{\mathrm{d}^{2n}}{\mathrm{d}\alpha^{2n}}\Big\{\frac{1}{2}+\varphi(\mathrm{e}^{2i\alpha})\Big\} \xrightarrow[\alpha\to\frac{\pi}{4}]{} 0.$$

Recall that the Jacobi theta function defined by

$$\theta(x) = \sum_{n\in\mathbb{Z}}\mathrm{e}^{-n^2\pi x} = 2\varphi(x)+1$$

satisfies the functional equation (Theorem 1.5.3)

$$\theta\Big(\frac{1}{x}\Big) = \sqrt{x}\,\theta(x).$$

This gives at once the functional equation

$$x^{-\frac{1}{4}} - 2x^{\frac{1}{4}}\varphi(x) = x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}\varphi\Big(\frac{1}{x}\Big),$$

or

$$\varphi(x) = \sqrt{x}\,\varphi\Big(\frac{1}{x}\Big) + \frac{1}{2}\sqrt{x} - \frac{1}{2}.$$

Hence

$$\begin{aligned}
\varphi(i+\delta) &= \sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi(i+\delta)}\\
&= \sum_{n=1}^{\infty}(-1)^n\,\mathrm{e}^{-n^2\pi\delta}\\
&= 2\varphi(4\delta)-\varphi(\delta)\\
&= \frac{1}{\sqrt{\delta}}\varphi\Big(\frac{1}{4\delta}\Big) - \frac{1}{\sqrt{\delta}}\varphi\Big(\frac{1}{\delta}\Big) - \frac{1}{2}.
\end{aligned}$$

From this it is easy to see that $\dfrac{1}{2} + \varphi(x)$ and all its derivatives tend to zero as $x \to i$ along any path in the cone $|\arg(x - i)| < \frac{\pi}{2}$. This proves the claim. Therefore, letting $\alpha \to \frac{\pi}{4}$ in (3.6.3), we obtain

$$\lim_{\alpha \to \frac{\pi}{4}} \int_0^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh(\alpha t)\, dt = \frac{(-1)^n \pi \cos \frac{\pi}{8}}{2^{2n}}. \tag{3.6.4}$$

To prove that Ξ has infinite real zeros we will show that Ξ changes sign an infinite number of times. Assume for contradiction that $\Xi(t)$ doesn't change sign for $t \geqslant T$. By (3.6.4) we know that

$$\lim_{\alpha \to \frac{\pi}{4}} \int_T^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh(\alpha t)\, dt = L,$$

for some $L \in \mathbb{R}$. Without loss of generality suppose that $\Xi(t) > 0$ for all $t \geqslant T$. Then

$$\int_T^{T'} \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh \alpha t\, dt \leqslant L$$

for all $\alpha < \frac{\pi}{4}$ and $T' > T$. Therefore, letting $\alpha \to \frac{\pi}{4}$ we get

$$\int_T^{T'} \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh \frac{\pi t}{4}\, dt \leqslant L.$$

Hence the integral

$$\int_T^{T'} \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh \alpha t\, dt$$

converges. This shows that the integral in (3.6.4) is uniformly convergent with respect to α for $0 \leqslant \alpha \leqslant \frac{\pi}{4}$, and it follows that

$$\int_0^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh \frac{\pi t}{4}\, dt = \frac{(-1)^n \pi \cos \frac{\pi}{8}}{2^{2n}},$$

for every $n \in \mathbb{N}$. This, however, is impossible; for, taking n odd, the right-hand side is negative, hence

$$\begin{aligned}\int_T^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh \frac{\pi t}{4}\, dt &< -\int_0^T \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh \frac{\pi t}{4}\, dt \\ &< KT^{2n},\end{aligned}$$

where K is independent of n. But by hypothesis there exists $m = m(T) > 0$ such that

$$\frac{\Xi(t)}{t^2 + \frac{1}{4}} \geqslant m,$$

for $2T \leqslant t \leqslant 2T + 1$. Therefore

$$\int_T^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh \frac{\pi t}{4}\, dt \geqslant \int_{2T}^{2T+1} m t^{2n}\, dt \geqslant m(2T)^{2n},$$

and from this it follows that

$$m 2^{2n} < K,$$

which is impossible for n sufficiently large. This completes the proof. $\square$

3.7 The Explicit Formula for $\psi(x)$

Recall that for $n \in \mathbb{N}$ we define the *von Mangoldt function* as

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some } p \in \mathbb{P} \text{ and for some } m \in \mathbb{N}^*; \\ 0, & \text{otherwise,} \end{cases} \tag{3.7.1}$$

and the *Chebyshev's ψ function* as

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n).$$

Note that ψ has jump discontinuities for prime powers. This leads us to consider the function ψ_0, that is the same at ψ except that at its jump discontinuities it takes the value halfway between the values to the left and the right, that is

$$\psi_0(x) := \begin{cases} \psi(x), & \text{if } x \neq p^m \text{ for some } p \in \mathbb{P} \text{ and for some } m \in \mathbb{N}^*; \\ \psi(x) - \frac{1}{2}\Lambda(x), & \text{otherwise.} \end{cases}.$$

The purpose of this section is to prove an explicit formula for ψ_0 that is related to the zeros of the Riemann zeta function, namely

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2} \log \Big(1 - \frac{1}{x^2}\Big),$$

where the sum over the nontrivial zeros $\rho = \beta + i\gamma$ of ζ is to be understood in the symmetric sense as

$$\lim_{T \to \infty} \sum_{|\gamma| \leqslant T} \frac{x^\rho}{\rho}.$$

This formula was conjectured by Riemann and proved by von Mangoldt in 1895. For $y > 0$, let

$$\delta(y) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{y^s}{s} \, \mathrm{d}s = \begin{cases} 0, & \text{if } 0 < y < 1; \\ \frac{1}{2}, & \text{if } y = 1 \\ 1, & \text{otherwise.} \end{cases}, \tag{3.7.2}$$

where $\alpha > 1$. Recall that (see (3.3.3))

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad (\mathfrak{Re}(s) > 1).$$

Letting $y = \frac{x}{n}$ in (3.7.2) and summing over $n \leqslant 1$ we get

$$\psi_0(x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \Big[-\frac{\zeta'(s)}{\zeta(s)} \Big] \frac{x^s}{s} \, \mathrm{d}s.$$

The idea of the proof is to move the vertical line of integration away to infinity on the left, and then express $\psi_0(x)$ as the sum of the residues of the function

$$\Big[-\frac{\zeta'(s)}{\zeta(s)} \Big] \frac{x^s}{s}$$

at its poles. The pole of $\zeta(s)$ at $s = 1$ contributes x; the pole of $\frac{1}{s}$ at $s = 0$ contributes

$$-\frac{\zeta'(0)}{\zeta(0)} = -\log 2\pi;$$

the trivial zeros of $\zeta(s)$ at $s = -2, -4, \ldots$ contributes

$$-\sum_{n=1}^{\infty} \frac{x^{-2n}}{-2n} = -\frac{1}{2} \log \Big(1 - \frac{1}{x^2}\Big),$$

and the nontrivial zeros ρ of ζ contributes

$$-\sum_{\rho} \frac{x^\rho}{\rho}.$$

To carry out this proof, we begin by proving a technical lemma which is known as *Perron's formula.*

Lemma 3.7.1. *For $y > 0$ let $\delta(y)$ be defined as in (3.7.2), and for $T \geqslant x > 0$ let*

$$\delta(y, T) := \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{y^s}{s} \, \mathrm{d}s.$$

If $y \neq 1$ then

$$|\delta(y) - \delta(y, T)| \leqslant y^\alpha \min \Big\{1, \frac{1}{T|\log y|}\Big\} \tag{3.7.3}$$

and for $y = 1$

$$|\delta(1) - \delta(1, T)| \leqslant \frac{\alpha}{T}.$$

Proof. First suppose that $0 < y < 1$. Consider the positively oriented rectangle with vertices at $R - it, R + it, \alpha + iT, \alpha - iT$. Since

$$\frac{y^s}{s} \xrightarrow[\sigma \to \infty]{} 0,$$

letting $R \to \infty$, by Cauchy's theorem,

$$\delta(y, T) = -\frac{1}{2\pi i} \int_{\alpha + iT}^{\infty + iT} \frac{y^s}{s} \, \mathrm{d}s + \frac{1}{2\pi i} \int_{\alpha - iT}^{\infty - iT} \frac{y^s}{s} \, \mathrm{d}s.$$

Now,

$$\Big| \int_{\alpha + iT}^{\infty + iT} \frac{y^s}{s} \, \mathrm{d}s \Big| \leqslant \frac{1}{T} \int_{\alpha}^{\infty} y^\sigma \, \mathrm{d}\sigma = \frac{y^\alpha}{T|\log y|},$$

and similarly for the other integral. This shows that

$$|\delta(y, T)| \leqslant \frac{y^\alpha}{T|\log y|}.$$

Next we use Cauchy's theorem to move the vertical line of integration to the right arc of the circle $|s| = |\alpha + iT|$ (see the figure below).

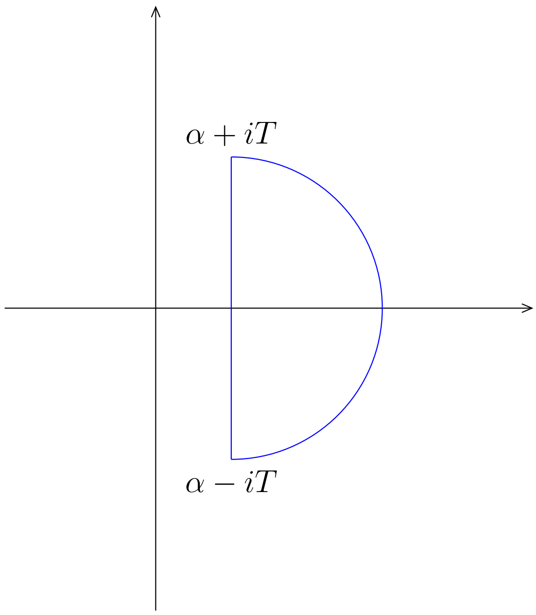

Figure 3.5: Semicircular contour

On the circular arc we have $|y^s| \leqslant y^\alpha$. Hence

$$|\delta(y, T)| \leqslant y^\alpha.$$

Combining both estimates we get (3.7.3) for $0 < y < 1$. The case $y > 1$ is similar but we move the integration using a rectangle or circular arc to the left. The contour then includes the pole at $s = 0$, where the residue is $1 = \delta(y)$. It remains the case $y = 1$, which is treated by direct computation. We have

$$\begin{aligned}\delta(1, T) &= \frac{1}{2\pi i}\big[\log(\alpha + iT) - \log(\alpha - iT)\big] \\ &= \frac{1}{2\pi}\big[\arg(\alpha + iT) - \arg(\alpha - iT)\big] \\ &= \frac{1}{\pi}\arg(\alpha + iT) \\ &= \frac{1}{2} + \frac{1}{\pi}\arg(T - i\alpha)\end{aligned}$$

Therefore

$$|\delta(1) - \delta(1, T)| \leqslant \arg(T - i\alpha) \leqslant \frac{\alpha}{T}.$$

This completes the proof $\square$

Now, for $x \geqslant 2$ let

$$\psi(x, T) := \frac{1}{2\pi i}\int_{\alpha - iT}^{\alpha + iT}\Big[-\frac{\zeta'(s)}{\zeta(s)}\Big]\frac{x^s}{s}\,\mathrm{d}s.$$

Then, by the previous lemma we have

$$|\psi_0(x) - \psi(x, T)| \leqslant \sum_{n \leqslant x}\Lambda(n)\Big(\frac{x}{n}\Big)^\alpha \min\Big\{1, \frac{1}{T|\log\frac{x}{n}|}\Big\} + \frac{c}{T}\Lambda(x), \tag{3.7.4}$$

where the term containing $\Lambda(x)$ is present only if x is a prime power. Now we proceed to estimate the series on the right of (3.7.4). Since $\alpha > 1$ is arbitrarily we may choose

$$\alpha = 1 + \frac{1}{\log x}.$$

Note that $x^\alpha = \mathrm{e}\,x$. We first take the terms with $n \leqslant \frac{3}{4}x$ or $n \geqslant \frac{5}{4}x$. For these terms,

$$\log \frac{x}{n} \ll 1.$$

Note that as $\alpha \to 1$, $x \to \infty$. Thus

$$-\frac{\zeta'(\alpha)}{\zeta(\alpha)} \ll \frac{1}{\alpha - 1} = \log x \qquad (x \to \infty)$$

Hence, the contribution of these terms to the sum is

$$\ll \frac{x}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\alpha} = \frac{x}{T}\Big[-\frac{\zeta'(\alpha)}{\zeta(\alpha)}\Big] \ll \frac{x}{T} \log x.$$

Now we consider the terms for which $\frac{3}{4}x < n < x$. Let q be the largest prime power less than x; we can suppose that $\frac{3}{4}x < n < x$, since otherwise the terms under consideration vanish. For the term $n = q$, since

$$\log(1+t) \leqslant t \qquad (t \geqslant 0)$$

we have

$$\log \frac{x}{n} = -\log\left(1 - \frac{x-q}{x}\right) \geqslant \frac{x-q}{x}.$$

Therefore the contribution of this term is

$$\ll \Lambda(q)\Big(\frac{x}{n}\Big)^\alpha \min\Big\{1, \frac{x}{T(x-q)}\Big\} \ll \log x \min\Big\{1, \frac{x}{T(x-q)}\Big\}.$$

For the other terms, we can put $n = q - \tau$, where $0 < \tau < \frac{x}{4}$. Then

$$\log \frac{x}{n} \geqslant \log \frac{q}{n} = -\log\left(1 - \frac{\tau}{q}\right) \geqslant \frac{\tau}{q}.$$

Hence the contributions of this terms is

$$\ll \sum_{0<\tau<\frac{x}{4}} \Lambda(q-\tau)\frac{q}{\tau T} \ll \frac{x}{T}\log^2 x.$$

The terms with $x < n < \frac{5}{4}x$ are dealt similarly, except that q is replaced by $\tilde{q}$, the least prime power greater than x. Let $\langle x \rangle$ denote the distance from x to the nearest prime power, other than x itself. Then combining all the obtained estimates we get

$$|\psi_0(x) - \psi(x,T)| \ll \frac{x\log^2 x}{T} + \log x \min\Big\{1, \frac{x}{T\langle x\rangle}\Big\}. \tag{3.7.5}$$

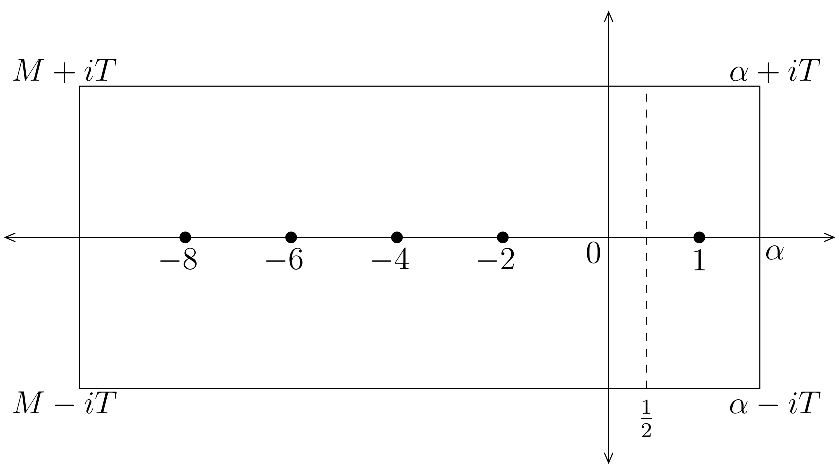

Figure 3.6: Contour of integration

Now consider the positively oriented rectangle with vertices at $\alpha-iT, \alpha+iT, -M+iT, -M-iT$, where M is a large negative odd integer, as shown in Figure 3.6. As discussed previously the sum of the residues of the integrand inside the rectangle is

$$x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \log 2\pi - \sum_{0<2n<M} \frac{x^{-2n}}{-2n}.$$

To estimate the horizontal integrals, by Lemma 3.4.4 we know that

$$N(T+1) - N(T) \ll \log T.$$

Hence, we can choose T so that no zero ρ of ζ has height near T. Specifically we may require

$$|T-\gamma| \gg \frac{1}{\log T}.$$

Recall further that for $s = \sigma + iT$, $-1 \leqslant \sigma \leqslant 2$ we have (see Theorem 3.4.2)

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|T-\gamma|<1} \frac{1}{s-\rho} + \mathcal{O}(\log T) \qquad (T \to \infty),$$

where $\rho = \beta + i\gamma$ runs over the nontrivial zeros of ζ. With our choice of T, each term of the sum is $\mathcal{O}(\log t)$ and by Lemma 3.4.4 the number of terms is also $\mathcal{O}(\log t)$. Thus

$$\frac{\zeta'(s)}{\zeta(s)} \ll \log^2 |s| \tag{3.7.6}$$

for s on such segment. To extend the previous bound for $\sigma \leqslant -1$ we use the functional equation (3.2.3) in the form

$$\zeta(1-s) = 2^{1-s}\pi^{-s} \cos\left(\frac{s\pi}{2}\right)\Gamma(s)\zeta(s).$$

We shall use this formula since, if $1-\sigma \leqslant -1$ the functions on the right have to be considered only for $\sigma \geqslant 2$. Therefore by Stirling's formula (2.2.13) it follows that

$$\frac{\zeta'(1-s)}{\zeta(1-s)} \ll \log 2|1-s| \qquad (\sigma \geqslant 2).$$

This together with (3.7.6) shows that for all $\sigma \in \mathbb{R}$ we have

$$\frac{\zeta'(s)}{\zeta(s)} \ll \log^2 |s|. \tag{3.7.7}$$

Using (3.7.7) we find that the contribution of the horizontal integrals is

$$\ll \int_{-\infty}^{\alpha} \frac{x^\sigma}{T} \log^2(|\sigma| + T)\, \mathrm{d}\sigma \ll \frac{\log^2 T}{T} \int_{-\infty}^{\alpha} x^\sigma \, \mathrm{d}\sigma \ll \frac{x \log^2 T}{T \log x}. \tag{3.7.8}$$

The contribution of the vertical integral is

$$\ll \int_{-T}^{T} \frac{\log^2 T}{M} x^{-M} \, \mathrm{d}t \ll \frac{T \log^2 T}{M x^M} \xrightarrow[M\to\infty]{} 0.$$

Thereby if we add the estimate in (3.7.8) to that in (3.7.5) we conclude

Theorem 3.7.1. *For $x \geqslant 2$ and for $T > 0$ we have*

$$\psi_0(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) + R(x,T), \tag{3.7.9}$$

where $\rho = \beta + i\gamma$ runs over the nontrivial zeros of ζ and

$$R(x,T) \ll \frac{x\log^2(xT)}{T} + \log x \min\left\{1, \frac{x}{T\langle x\rangle}\right\}. \tag{3.7.10}$$

Letting $T \to \infty$ in the previous theorem finally gives us

Corollary 3.7.1.1 (The prime numbers formula)**.** *For $x \geqslant 2$ we have*

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right),$$

where the series over the nontrivial zeros $\rho = \beta + i\gamma$ of ζ is evaluated as the limit

$$\lim_{T\to\infty} \sum_{|\gamma|<T} \frac{x^\rho}{\rho}.$$

Note that if $x \in \mathbb{Z}$ then $\langle x\rangle \geqslant 1$ and hence (3.7.10) takes the simpler form

$$|R(x,T)| \ll \frac{x}{T}\log^2(xT). \tag{3.7.11}$$

3.7.1 The Zero-free Region and the PNT

In 1896 Jacqued Hadamard and de la Vallée Poussin proved independently that if $\pi(x)$ denotes the number of primes less than or equal to x then

$$\pi(x) \sim \frac{x}{\log x} \qquad (x \to \infty).$$

This result is known as the prime number theorem (PNT). Recall that in Theorem 2.4.4 we showed that the PNT is equivalent to

$$\psi(x) \sim x \qquad (x \to \infty).$$

Using the results of the previous section we shall now deduce that there exists $c > 0$ such that

$$\psi(x) = x + \mathcal{O}\{x\exp(-c\sqrt{\log x})\}$$

For doing so we need the following lemma

Lemma 3.7.2. *Let $s = \sigma + it$. Then, there exists $c > 0$ such that*

$$\zeta(s) \neq 0 \qquad \textit{for} \quad \sigma \geqslant 1 - \frac{c}{\log t}.$$

Proof. As usual, let $s = \sigma + it$ and $\rho = \beta + i\gamma$ be a zero of ζ. For $\sigma > 1$ we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Using the familiar inequality

$$3 + 4\cos\theta + 2\cos 2\theta = (1 + 2\cos\theta)^2 \geqslant 0,$$

and following the same line of argument as in the proof of Theorem 3.3.1 it is easy to see that

$$3\Big[-\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big] + 4\Big[-\mathfrak{Re}\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\Big] + 2\Big[-\mathfrak{Re}\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\Big] \geqslant 0. \tag{3.7.12}$$

Since $\zeta(s)$ has a simple pole at $s = 1$ then

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} = \frac{1}{\sigma - 1} + \mathcal{O}(1).$$

Using Theorem 3.4.2 we get

$$\mathfrak{Re} - \frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} < A\log t - \mathfrak{Re}\frac{1}{\sigma + it - \rho},$$

and

$$\mathfrak{Re} - \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \ll \log t.$$

Taking $t = \gamma$ we have

$$\mathfrak{Re}\frac{1}{\sigma + it - \rho} = \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2} = \frac{1}{\sigma - \beta}.$$

Inserting these results to (3.7.12) we get

$$\frac{4}{\sigma - \beta} < \frac{3}{\sigma - 1} + A\log t.$$

Take

$$\sigma = 1 + \frac{\delta}{\log t},$$

where $\delta > 0$ is small. Then

$$\beta < 1 + \frac{\delta}{\log t} - \frac{4\delta}{(3 + A\delta)\log t}.$$

Since $4 > 3$ we can choose $\delta > 0$ such that

$$\beta < 1 - \frac{c}{\log t},$$

for some $c > 0$. This completes the proof. □

Now we are ready to prove

Theorem 3.7.2. *For $x \geqslant 2$ we have*

$$\psi(x) = x + \mathcal{O}\big\{x\exp(-c\sqrt{\log x})\big\}. \tag{3.7.13}$$

Proof. By the previous lemma there exists c such that if $|\gamma| < T$ then

$$\beta < 1 - \frac{c}{\log T}.$$

Therefore

$$|x^\rho| = |x^\beta| \leqslant x^{1-\frac{c}{\log T}}$$

Now, since (see (3.4.8))

$$N(T+1) - N(T) \ll \log T,$$

then

$$\sum_{|\gamma|<T} \frac{1}{|\rho|} \ll (\log T)^2.$$

Therefore

$$\sum_{|\gamma|<T} \left|\frac{x^\rho}{\rho}\right| \ll x(\log^2 T)x^{-\frac{c}{\log T}}.$$

Without loss of generality we can assume that $x \in \mathbb{Z}$. Hence using Theorem 3.7.1 and the form of $R(x,T)$ as in (3.7.11) we obtain

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} + \mathcal{O}\Big(\frac{x}{T}\log^2(xT)\Big)$$

Thereby we get

$$|\psi(x) - x| \ll x\log^2(xT)\Big\{x^{-\frac{x}{\log T}} + \frac{1}{T}\Big\}$$

Taking $T = \exp c\sqrt{\log x}$ we get (3.7.13) with a different constant. $\square$

Letting $x \to \infty$ in (3.7.13) it immediately follows

Corollary 3.7.2.1 (The Prime Number Theorem)**.** *Let $\pi(x)$ denote the number of primes less than or equal to x. Then*

$$\pi(x) \sim \frac{x}{\log x}$$

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