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# Introduction to Geometric Group Theory

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Degree work submitted to the Department of Mathematics to qualify for the degree of Mathematics

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Bogotá - Colombia
October, 2019

# Abstract

The purpose of this document is to introduce concepts of an area of mathematics known as geometric group theory which develops the study of finitely generated groups, exploring the connection between the algebraic properties of these with geometric and topological properties of spaces they act on.

The document consists of three chapters grouped as two parts, the first part is composed of chapters one and two, where the objective is to give groups a representation as metric spaces and give them geometric properties, such as metrics, geodesics, paths, etc. For this, we use important tools as Cayley's Graphs and growth functions. Also, we study two explicit examples which are the Lamplighter Group $\mathcal{L}_2$ and the Thompson's Group $\mathcal{F}$, where the potential of these tools in the study of infinite groups, can be evidenced.
The second part (third chapter), makes an introduction to a very interesting relation between metric spaces known as quasi-isometries and Švarc-Milnor Lemma, that uses the given concepts in the first part to relate finitely generated groups with metric spaces, giving also important properties and ideas to classify this groups up to quasi-isometries.

# Acknowledgments

First of all and as always, I thank God for allow me to discover such a beautiful area of knowledge as the mathematics. To my parents and my siblings, that have been supporting me for this years. To all of my professors, specially Mario, that have teach me the beauty and toughness of mathematics. Finally to my friends, Juanita, Gaitan, Thomas, Sebastian, Diana and all of the other many that I can not mention, each one of you have been important in my life and this work couldn't have been completed without your help.
Thanks for everything.

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# Introduction

The notion of groups is one of the most important ideas in mathematics, it entails a big variety of mathematical objects, properties and tools that have been studied for years. Since the 1700s Lagrange and Vandermonde were discovering different properties of permutations while studying the solution of equations by radicals. Years later Galois understood "group" as the group of permutations of a finite set giving the first concepts of this theory. Galois work was published only in 1846, fourteen years after Galois's dead, his work was taken and systematized by Cauchy that was the first to o consider the possibility of more abstract group elements.
In 1854 Arthur Cayley gave a first approximation of the theorem that years later was going to be named in his honor, that declare that every group is a subgroup of a permutation group. He also in 1878 the concept of Cayley's graphs (that were reintroduced in 1909 by Max Dehn under the name of Grouppenbild that means group diagram), this idea led to the geometric group theory of today.

With finite groups the existence of generators and relations was easy and not interesting to solve, the real problem rises when we ask if it is possible to find sets of generators and relations for infinite groups, this problem was solved by Felix Klein's student, which lead the foundation of the geometric group theory, or how it was introduced in the 1880s, combinatorial group theory.
In the first half of the 20th century many different mathematicians introduce topological and geometric ideas outside the traditional combinatorial tools into the study of discrete groups, but the emergence of geometric group theory as a new area was given in the late 1980s when Mikhail Gromov introduced the notion of hyperbolic groups in his essay "*Hyperbolic Groups*" in 1987, and his subsequent monograph "*Asymptotic Invariants of Infinite Groups*", where captures the ideas of a finitely generated group to have a large-scale negative curvature and the concept of quasi-isometries, a large-scale relation between metric spaces that was used to see geometric properties on groups, an idea completely revolutionary.

After this many themes and developments have been done, as the study of Dehn's functions, the interactions with computer science, complexity theory, theory of formal languages, measure-theoretic properties of group actions on metric spaces, new methods on group cohomology, etc.

# Chapter 1

# The Cayley Graph

In this chapter we will introduce a notion of a Cayley graph and construct some examples. The Cayley graphs were introduced by Arthur Cayley in the late 1870s, they are a useful tools in algebra, combinatorics and other areas, in particular we are going to introduce them as an algebraic structure and in the following chapters we will see that they can be seen as metric spaces and a way to relate the groups that they represent with some metric spaces. Also in the second part of this chapter we will construct an interesting example known as the Lamplighter group $\mathcal{L}_2$

## 1.1 Cayley Graphs

**Definition 1.1.1.** A graph $\Gamma$ consist of a pair $(V(\Gamma), E(\Gamma))$ of vertices and edges respectively where each edge is associated to a pair of vertices. If for two vertices $\{u, v\}$ there exist an edge that is associated to both, we say that $u$ and $v$ are adjacent.

This definition can be complemented by adding other characteristics as:

1. Locally finite: If each vertex is contained in a finite number of edges.
2. Labeled: It can be *vertex labeled* or *edge labeled* if each element of $V(\Gamma)$, or $E(\Gamma)$, respectively, is labeled.
3. Connected: If for each pair of vertices $\{u, w\}$ there exist a sequence of vertices and edges, $\{u = v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n = w\}$ where $\{v_i, v_{i+1}\}$ are adjacent for each $i$ (this sequence is a path in $\Gamma$).
4. Directed: For each edge it is defined an initial vertex and a terminal vertex. Graphically this direction is often indicated as an arrow.
5. Decorated: Have different elements such as , directed edges, labeled or colored vertices and/or edges, etc.

**Definition 1.1.2.** If $X$ is a set, we will denote by $\text{Sym}(X)$ the collection of all bijections from $X$ to $X$ that preserve the indicated mathematical structure. For example, if $X$ is a graph, $\text{Sym}(X)$ is the bijections of X that preserve the structure of the graph as vertex and edges.

Note that under the composition $\text{Sym}(X)$ is a group, for example if we consider $X$ to be a graph $\Gamma$, then $\text{Sym}(\Gamma)$ consists of all the bijections $\tau$ taking vertices to vertices and edges to edges, such that if $e \in E(V)$ with ending vertices $v, w$, then the ending vertices of $\tau(e)$ are $\tau(v), \tau(w)$. In particular the symmetry group of a decorated graph is the collection of all the symmetries that preserve all the decorations. This is going to be explain in 1.1.12.

**Definition 1.1.3.** An action of a group $G$ on a set X (in our case a graph) is a group homomorphism from $G$ to $\text{Sym}(X)$, equivalently, it can be defined as a map from $G \times X \to X$ that satisfies the following two axioms:

1. $e \cdot x = x$, for all $x \in X$;

2. $(gh) \cdot x = g \cdot (h \cdot x)$, for all $g, h \in G, x \in X$,

And it is denoted as "$G$ acts on X" by $G \curvearrowright X$.

If we have a group action $G \curvearrowright X$ then the associated homomorphism is a *representation* of $G$, and it is said to be *faithful* if this homomorphism is injective.

**Theorem 1.1.4.** *Every finitely generated group can be faithfully represented as a group of permutations.*

*Proof.* The proof of this theorem constructs a representation of $G$ as a group of permutations of itself, and it is a standard theorem in a course of abstract algebra. The proof can be found in Corollary 4.6 of [10]. $\square$

An important aspect of the use of this theorem is the action of the group, this here there is a construction of a representation of $G$ as a group of permutations on itself, we keep this in mind the whole chapter.

**Theorem 1.1.5.** *Every finitely generated group $G$ can be faithfully represented as a symmetry group of a connected, directed, locally finite graph.*

*Proof.* Let $G$ be a finitely generated group with generating set $S = \{s_1, \ldots, s_n\}$. We can prove this theorem by constructing a graph, $\Gamma_{G,S}$ on which $G$ acts. The vertices of $\Gamma_{G,S}$ are the elements of $G$. For each $g \in G$, $s \in S$, make an edge labeled $s$ from the vertex labeled $g$ to the vertex labeled $gs$. Since $G$ is finitely generated, this graph is locally finite. Since $S$ generates $G$, this graph is connected.

By construction, $\Gamma_{G,S}$ is directed. Let $G$ act on the graph by left multiplication, that is, for any $g \in G$, $g$ will send the vertex labeled $h$ to the vertex labeled $gh$.

This action can be extended to an action on the edges (see figure 1.1). The vertex $v_h$ is joined to $v_{hs}$ via the edge (generator) $s$; by the action of $g$, $v_h$ goes to $v_{gh}$ and $v_{hs}$ to $v_{ghs}$, so we can define the action on the edge labeled $s$ that joins $v_h$ to $v_{hs}$ sending it to the edge labeled also $s$ joining $v_{gh}$ to $v_{ghs}$. Note that even if the action is defined on the left, for the edges it is given in terms of right multiplication.

$\square$

Keeping this in mind, the graph that we constructed in the last theorem that represents $G$ is known as the Cayley graph of $G$.

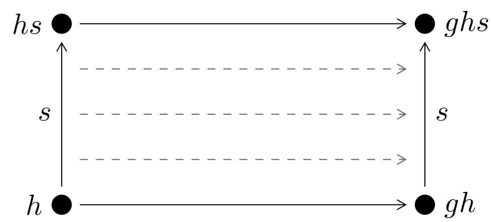



Figure 1.1: Action on the vertices can be extended to edges.

**Definition 1.1.6.** Let $G$ be a finitely generated group and $S$ the generating set, we can define the Cayley graph $\Gamma_{G,S}$, which is a directed graph that can be constructed following the next steps:

1. Each $g \in G$ is a vertex of $v_g \in V(\Gamma_{G,S})$
2. Each $s \in S$ forms a directed edge with initial vertex $v_g$ and terminal vertex $v_{g \cdot s}$; Giving each edge a correspondence with right multiplications of the elements of $S$.

In general, for different generators $s_1$ and $s_2$ it can be assigned colors $c_1$ and $c_2$ respectively to differentiate the action of the different elements of $S$. After understanding the definition, an easy way to understand the behavior of Cayley graph is to make some examples.

**Example 1.1.7.** The first and a very intuitive example is to consider the cyclic group of $n$ elements, $C_n$, clearly 1 is a generator of $C_n$, so the only edges are given by the action of $1 \cdot g$ for $g \in C_n$, then the Cayley graph is illustrated at figure 1.2.

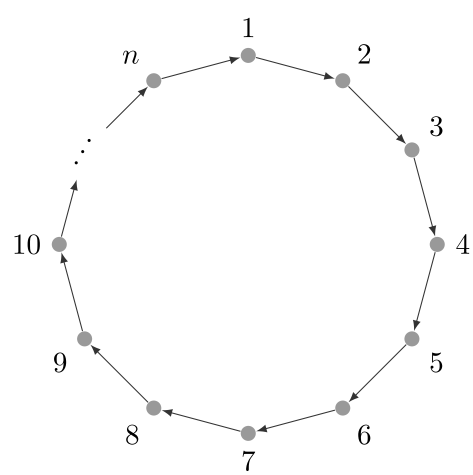



Figure 1.2: Cayley graph of the cyclic group with $n$ elements, $C_n$.

**Example 1.1.8.** For an explicit and harder example of the construction of a Cayley graph, we will consider $G = A_4$, the alternating group that consists of the twelve even permutations of $\{1,2,3,4\}$ taking $S = \{(123), (12)(34)\}$.

Let us calculate the action of $(123)$ over the elements of $G$:

- $(123) \cdot (123) = (132)$
- $(123) \cdot (132) = e$
- $(123) \cdot (12)(34) = (243)$...

Then we can calculate the action of $(12)(34)$ over $G$:

- $(12)(34) \cdot (123) = (134)$
- $(12)(34) \cdot (12)(34) = e$...

After knowing the relation of these elements, it is easy to construct the Cayley graph given in the Figure 1.3, where the dashed lines represent the action of $(12)(34)$ and the others the action of $(123)$.

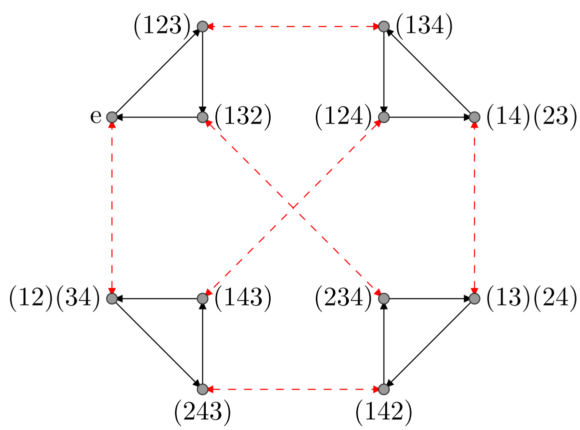



Figure 1.3: Cayley graph of $A_4$.

In the definition of Cayley graph, we don't have a restriction on $|G|$, so we can consider Cayley graphs of infinite groups.

**Example 1.1.9.** The Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$ with respect to the generators $\{(1,0),(0,1)\}$ appears in the Figure 1.4.

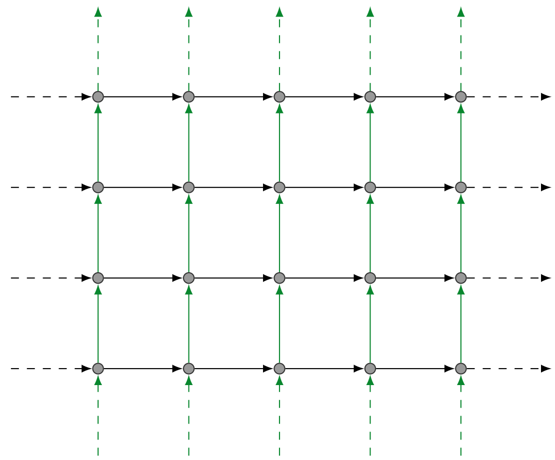


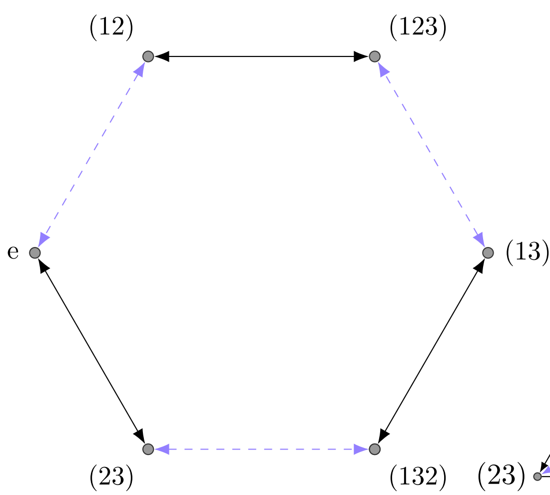
Figure 1.4: Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$.

**Remark 1.1.10.** An important remark is that the Cayley graph depends on the generating set $S$, so it is not unique, as the following example shows

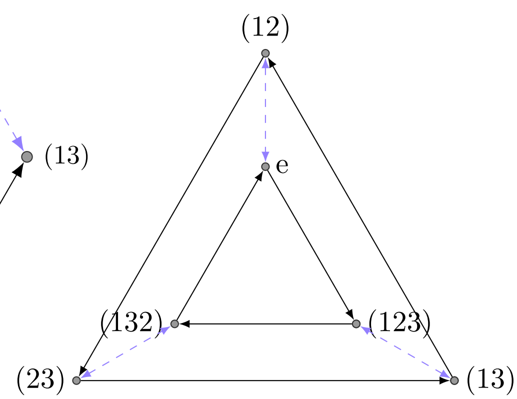
**Example 1.1.11.** Consider $G = S_3$, with $S = \{(12), (123)\}$, and on the other hand with $S = \{(12), (23)\}$. The first one is illustrated in Figure 1.5b, and the other in Figure 1.5a. In those graphs, to avoid the use of cycles, we will use the double arrows to simplify the graph.

**Definition 1.1.12.** Let $\Gamma$, a decorated graph, we define $\mathrm{Sym}^+(\Gamma)$ the subgroup of $\mathrm{Sym}(\Gamma)$ that preserves all of the declared decorations.

The following theorem indicates that all finitely generated groups can be realized as symmetries (that preserve labels and orientation) of locally finite directed graphs. This is important because in the following chapters we are going to use the representation of $G$ as the Cayley graph more than the group itself.




(a) C-G of $S_3$ with $S = \{(12), (23)\}$.




(b) C-G of $S_3$ with $S = \{(12), (123)\}$.

Figure 1.5: Cayley graphs of $S_3$ with different generators

**Theorem 1.1.13.** *Let $\Gamma_{G,S}$ be the Cayley graph of a group $G$ with respect to a finitely generating set $S$. Consider $\Gamma_{G,S}$ to be decorated with directions on its edges and labeling of its edges, corresponding to the generating set $S$. Then $G \cong \mathrm{Sym}^+(\Gamma_{G,S})$.*

*Proof.* Let's consider the action of $G$ on $\Gamma_{G,S}$ by translation (the same used in 1.1.5, that also shows that the representation is faithful). Since this is a left action, we have shown that this does not affects the direction or the labelling of the edges (the action on the edges is a right action on 1.1.5), therefore the representation is faithful into $\mathrm{Sym}^+(\Gamma_{G,S})$.
Now we need to show the surjectivity. For this, consider an arbitrary element $\tau \in \mathrm{Sym}^+(\Gamma_{G,S})$ and we will construct a preimage. For any $g \in G$, let $v_g$ the vertex in $\Gamma_{G,S}$ corresponding to $g$. There exist a $g$ such that $\tau(v_e) = v_g$. If we consider $g$ as a symmetry of $\Gamma_{G,S}$, the product $\tau \cdot g^{-1} \in \mathrm{Sym}^+(\Gamma_{G,S})$ and also $v_e \mapsto v_e$ with this symmetry, further, it fixes all edges arriving at or leaving from $v_e$. As an element of $\mathrm{Sym}^+(\Gamma_{G,S})$ it fixes all the vertices adjacent to $v_e$ and again their edges, so the symmetry $\tau g^{-1}$ is the identity, and because of that, $\tau = g$. All this says that the preimage of a symmetry is only determined by what it does to $v_e$. In this case $g$ is the preimage of $\tau$, and as this was for an arbitrary $\tau$, $\mathrm{Sym}^+(\Gamma_{G,S}) \cong G$. □

An illustrative example is to consider $G = C_4$ with generating set given by the element $\{1\}$. It is easy to see that the Cayley graph is the figure 1.6.

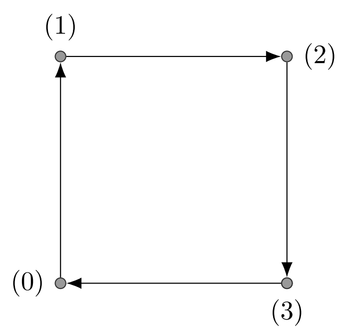



Figure 1.6: Cayley graph of $C_4$

So, having this, it is easy to see that after a rotation of 90 degrees:

| vertex | edges |
|---|---|
| $1 \longmapsto 2$ | $1 \longmapsto 2$ |
| $2 \longmapsto 3$ | $2 \longmapsto 3$ |
| $3 \longmapsto 0$ | $3 \longmapsto 0$ |
| $0 \longmapsto 1$ | $0 \longmapsto 1$ |

But after a reflection over the y-axis, we have a problem, the edge that starts on 1 and goes to 2, becomes an edge that starts on the position of 2 and ends on the position of 1, i.e

$$2 \longmapsto 1,$$

but this does not preserve the decorations of our graph, so even if that reflection is a symmetry of our graph, it does not contradict the theorem.

The symmetry of the rotation of 90° keeps the direction of the arrows, but the reflection over the vertical dashed line does not, and makes sense because $\mathrm{Sym}^+(\Gamma_{C_4,1}) \cong C_4$.

**Example 1.1.14.** Another interesting example is to consider the Petersen graph in Figure 1.7, it is easy to see that $\Gamma$ is vertex transitive (there exists a symmetry of the graph that translates a vertex to any other), just as a Cayley graph, but there is no group $G$ and generating set $S$ such that the Petersen graph is the underlying graph of the Cayley of $\Gamma_{G,S}$.

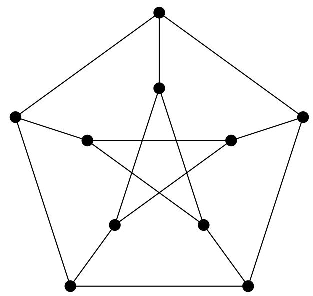


Figure 1.7: Petersen graph

A simple explanation is that if there is such a group, it has to have 10 elements, so there are only 2 options, $G = C_{10}$ or $G = D_5$ where $D_5$ is the dihedral. If we suppose that $G = C_{10}$, we can take any 2 generators $a, b$, then $a^{-1}b^{-1}ab = \mathrm{Id}$, which gives a cycle of length 4 in the graph, but Petersen graph has none of those. In a similar way, if $G = D_5$, then $(ab)^2 = Id$, that also is a cycle of length 4.

The following definition will be very important for the rest of the document, there are some different definitions of what a free group is, but we will use the one in [1].

**Definition 1.1.15.** Given a set $S = \{s_1, s_2, \ldots, s_n\}$ of elements in a group $G$, an element of the new group consists of a reduced word (this concept will be studied in the second chapter with Definition 2.1.2), i.e, we have canceled any adjacent pair of elements that are inverse to each other, using the elements of $S$ and $S^{-1}$, where $S^{-1}$ represents the set of formal inverses.

**Example 1.1.16.** Considering $S = \{a, b\}$, we define $\mathbb{F}_2$ as the free group with 2 elements. Note that the generators of $\mathbb{F}_2$ are $a$ and $b$, so is easy to see that $\Gamma_{\mathbb{F}_2,\{a,b\}}$, is an infinite tree, that locally looks like figure 1.8:

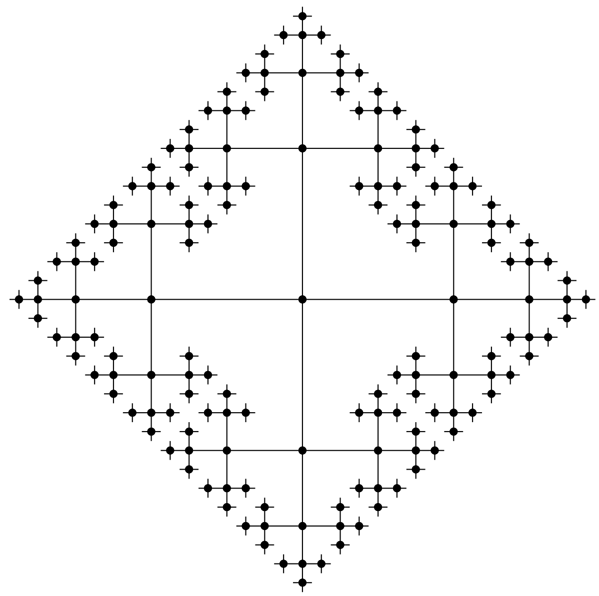


Figure 1.8: Local view of the Cayley Graph of $\mathbb{F}_2$

## 1.2 The Lamplighter Group $\mathcal{L}_2$

As we mentioned before, we want to construct an interesting example known as the Lamplighter group $\mathcal{L}_2$, according to [11] the name was given by James Cannon because of the interpretation of this. To construct this example we need some concepts before.

**Definition 1.2.1.** We will construct a group $G$ from other groups $H$ and $K$, let us consider

$$\begin{aligned} \phi: \quad & K \longrightarrow \mathrm{Aut}(H) \\ & k \longmapsto \phi_k, \end{aligned}$$

to be a group homomorphism from $K$ to the automorphism group of $H$. The elements of the associated semi-direct product are ordered pairs of elements $[h, k]$, where the operation is

$$[h_1, k_1] \cdot [h_2, k_2] = [h_1 \cdot \phi_{k_1}(h_2), k_1 \cdot k_2]\,.$$

Since $\phi_{k_1}(h_2) \in H$, $h_1 \cdot \phi_{k_1}(h_2)$ is computed in $H$ and $k_1 \cdot k_2$ in $K$. This product define a group that is known as the (outer) semidirect product of $H$ and $K$ and it is denoted $H \rtimes K$.

**Definition 1.2.2.** We start by forming a direct sum of copies of $G$ by elements of $h \in H$, indexing the sum by the elements of $H$ as $\bigoplus_{h \in H} G$, then the *wreath product*, denoted as $G \wr H$, is defined as:

$$G \wr H = \left( \bigoplus_{h \in H} G \right) \rtimes H.$$

The action of $H$ in the sum is defined as follows, given $\vec{g} \in \bigoplus_{h \in H} G$, the element $h \in H$ permutes the entries of $\vec{g}$ by taking the entry in position $h'$ to the position $h \cdot h'$ for every $h' \in H$.

**Example 1.2.3.** An easy example of this, is to consider $\mathbb{Z}_2 \wr \mathbb{Z}_3$. This is the semi-direct product $(\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_3$. If we declare $\phi_1 \in \mathrm{Aut}((\mathbb{Z}_2)^3)$ as the cyclic permutation $\phi_1(a,b,c) = (b,c,a)$, then we can see the behavior, for example computing:

$$\begin{aligned}
[(0,1,1),1]\cdot[(1,0,0),1] &= [(0,1,1)\cdot\phi_1(1,0,0), 1\cdot 1] \\
&= [(0,1,1)\cdot(0,0,1), 1+1] \\
&= [(0,1,1)\cdot(0,0,1), 2] \\
&= [(0,1,1)+(0,0,1), 2] \\
&= [(0,1,0), 2]\,.
\end{aligned}$$

The remainder of this section is devoted to understand the example $\mathbb{Z}_2 \wr \mathbb{Z}$. It is the semidirect product:

$$\mathbb{Z}_2 \wr \mathbb{Z} = \left(\bigoplus_{h\in H} \mathbb{Z}_2\right) \rtimes \mathbb{Z} = (\cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots) \rtimes \mathbb{Z}$$

This particular example is called the *lamplighter group*, and it is denoted $\mathcal{L}_2$. First of all we will give a geometric description of how this group can be understood (and also the reason of its name). Imagine a rural town with an infinite main street lined with lampposts, a lamplighter walks up and down the street lighting some of the light bulbs, then ends his walk just in front one of the lampposts. This situation is what $\mathcal{L}_2$ represents. To understand this analogy let us divide each part of this product:

- First of all, we have the group $\mathbb{Z}_2$, that from the description we gave above, we can think of this as a lamppost where the elements of $\mathbb{Z}_2$ represents if it is *"on"* or *"off"* More specifically if the element of $\mathbb{Z}_2$ is $[1]$, we say that the lamp is *"on"* and if the element is $[0]$, we say that it is *"off"*.
- $\bigoplus_{h\in H} \mathbb{Z}_2$ can be thought as a line of infinite lamps (copies of $\mathbb{Z}_2$) that are on or off depending on the elements of $\mathbb{Z}_2$.
- The elements of $\mathcal{L}_2$ are determined which (finite number) entries have non zero elements, in the analogy the elements are infinite lines of lamps where some of them are on.

In this group the identity element corresponds to $\vec{0} = (\dots,0,0,0,\dots) \in \mathcal{L}_2$. We need to understand the operation of $\bigoplus_{h\in H} \mathbb{Z}_2$, viewing elements of $\mathcal{L}_2$ as subsets $S \subset \mathbb{Z}$. As there can only be finite non zero elements, the operation corresponds to the symmetric difference ("$\triangle$"). For example if $S = \{-2,0,1\}$ and $T = \{-3,0,4\}$, then $\{-2,0,1\}\triangle\{-3,0,4\} = \{-3,-2,1,4\}$.

Understanding this, every element in $\mathcal{L}_2$ can be represented by an ordered pair $[S,n]$ where $S \subset \mathbb{Z}$ and $n \in \mathbb{Z}$, and defining the operation by:

$$[S,n]\cdot[T,m] = [S\triangle(T+n), n+m],$$

where $(T+n) = \{t+n | t \in T\}$. We denote the identity element as $[\varnothing, 0]$

**Lemma 1.2.4.** The lamplighter group $\mathcal{L}_2$ can be generated by two elements, one of order 2 and the other of infinite order.

*Proof.* Let $t$ be the element $[\varnothing, 1] \in \mathcal{L}_2$ and $a = [\{0\}, 0] \in \mathcal{L}_2$, notice that $a \neq [\varnothing, 0]$. The product of them is:

$$ta = [\varnothing, 1] \cdot [\{0\}, 0] = [\{1\}, 1],$$

more generally:

$$t^n a = [\{n\}, n]$$

and

$$t^n a = [\{n\}, n]$$

Doing some more computations, it is easy to prove that

$$\mathcal{L}_2 \ni [\{n_1, n_2, \ldots, n_m\}, k] = t^{n_1} a t^{-n_1} \cdot t^{n_2} a t^{-n_2} \cdots t^{n_m} a t^{-n_m} \cdot t^k.$$

Therefore, the set $\{a, t\}$ is a generating set for the lamplighter group. □

The figures 1.9 and 1.10 are visual representations of the elements of $\mathcal{L}_2$. For $\mathcal{L}_2 \ni [\{n_1, n_2, \ldots, n_m\}, k]$ we will color the vertices corresponding to $\{n_1, n_2, \ldots, n_m\}$ in yellow, and all the others with black; the 0 is marked with a line (to differentiate it); and finally adding a pointer pointing to the vertex associated to $k$.

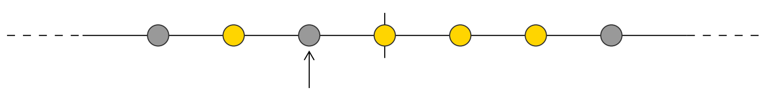


Figure 1.9: A geometric representation of $[\{-2, 0, 1, 2\}, -1] \in \mathcal{L}_2$

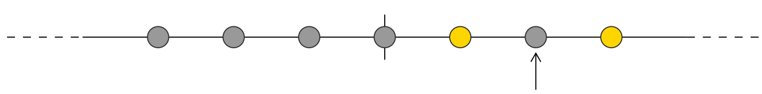


Figure 1.10: A geometric representation of $[\{3, 1\}, 2] = t^3 a t^{-2} a t \in \mathcal{L}_2$

If we want to study the Cayley graph with respect of the generating set $\{a, t\}$, we need to understand the effect of the right multiplication of $a$ and $t$. If $g = [S, k]$ is an arbitrary element of $\mathcal{L}_2$, then:

$$g \cdot a = [S \triangle \{k\}, k + 0] = [\widehat{S}, k],$$

where $\widehat{S}$ either adds or removes $k$ to $S$. In terms of the pictures of the elements, this is changing the color (”or” and ”off”) of the vertex that has the pointer pointing at.

Right multiplication by $t$ is:

$$g \cdot t = [S, k] \cdot [\varnothing, 1] = [S, k + 1].$$

This simply moves the pointer one unit to the right. Similarly right multiplication by $t^{-1}$ moves the pointer one unit to the left. This pictures are the reason why the group is called the *lamplighter* group. Thinking on a lamplighter that is stationed at a the position of the pointer, and turns on and off the lamps depending of the elements of $\mathbb{Z}_2 \wr \mathbb{Z}$.

An important thing to notice in order to try to draw the Cayley graph of $\mathcal{L}_2$ is that the element of Figure 1.10 has another representation, as $tat^2at^{-1}$, and that relation gives us a cycle in the Cayley graph. In Figure 1.11 we show a cycle in Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$ with

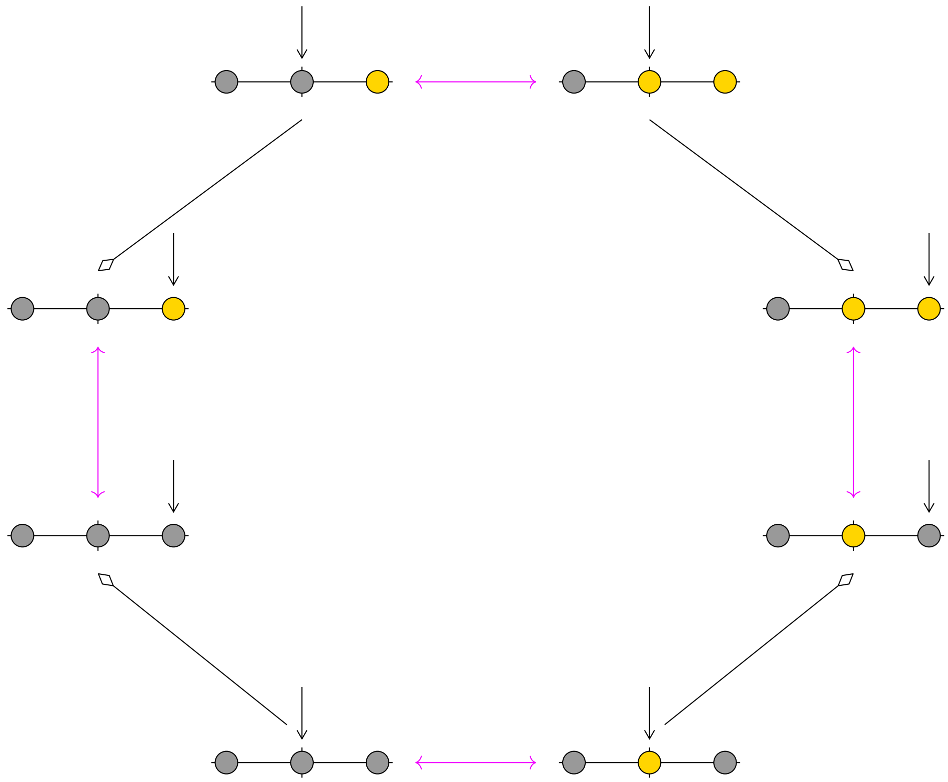


Figure 1.11: A cycle in the Cayley graph of the Lamplighter Group

generating set $S = \{a, t\}$ using black for $t$ and pink for $a$, this corresponds to the relation $atat^{-1} = tat^{-1}a$. An induction argument shows that $at^n at^{-n} = t^n at^{-n}a$.

The Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$ is fairly complicated, but Figure 1.12 shows a local view around the identity, using the explicit elements instead of the geometric representation and the same colors of the cycle example. Note that because of what we see at Figure 1.11 this graph is not a tree even if the local view looks as one.

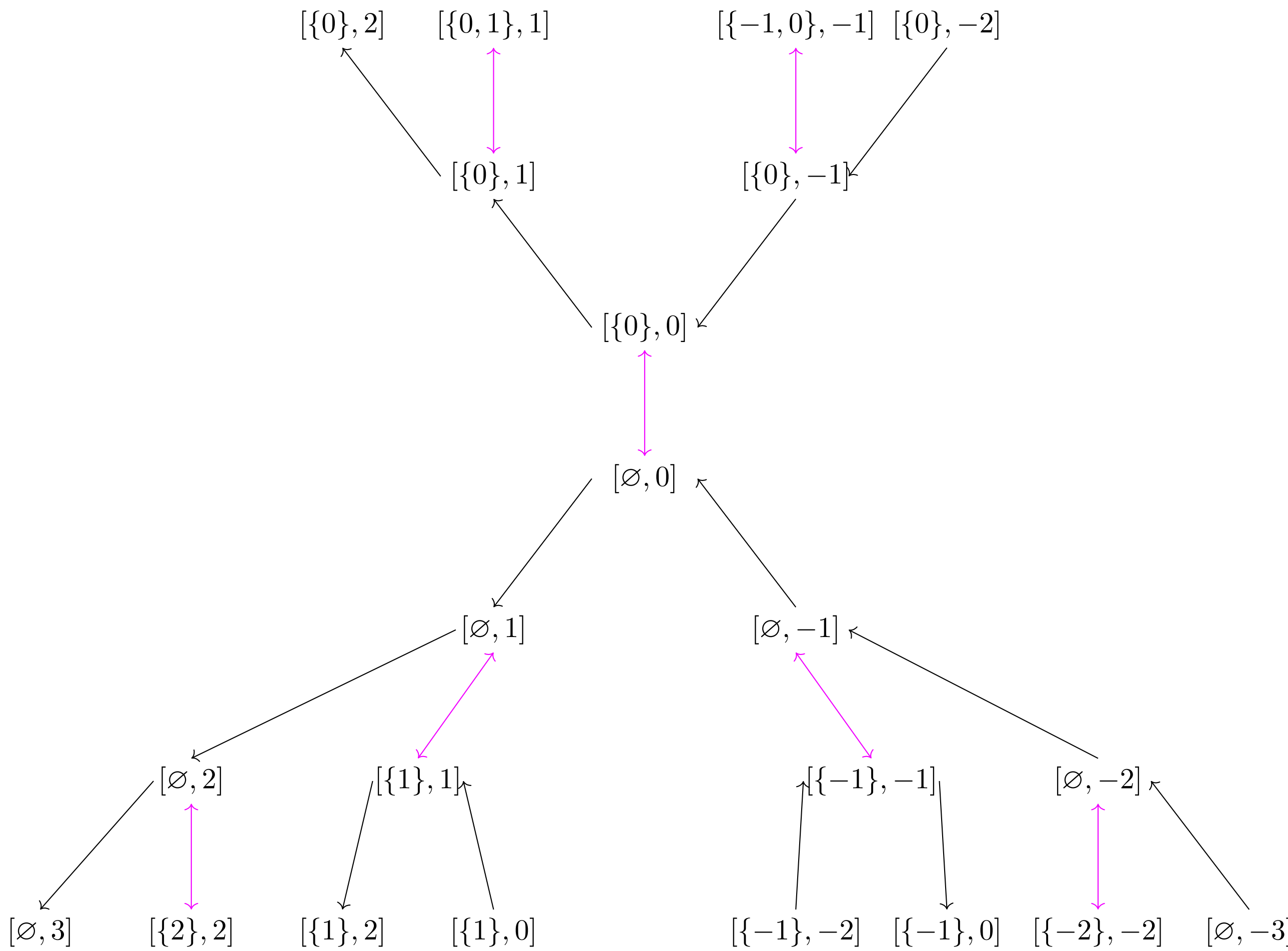



Figure 1.12: Local view of Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$

# Chapter 2

# Growth of Groups

An interesting question that appears in the study of infinite groups is how to give a comparison between the sizes of infinite groups. For example consider the free abelian $\mathbb{Z}^2$ and the free group $\mathbb{F}_2$, of course both groups have countably infinite cardinality, so how can we compare them? The answer can be found on the growth functions that we are going to define on the third part of this chapter.

We will divide this chapter into 3 parts, in the first one, before giving the definition of growth functions, we need some notions on the geometric properties of a group. In the second part we introduce the Thompson's Group $\mathcal{F}$, a very interesting group that seems to look like the answer to a group that has "intermediate growth", a concept that is finally introduced in the third part together with the definition of growth functions and some examples of this.

## 2.1 Geometric Concepts on a Group

**Definition 2.1.1.** (Metric space). A metric space, consist of a set $X$ and a distance function $d : X \times X \to \mathbb{R}$, such that, for ant $x, y, z \in X$

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ iff $x = y$ ,
3. $d(x, y) = d(y, x)$ ,
4. $d(x, y) + d(y, z) \geq d(x, z)$.

A function from a metric space to another, $\varphi : X_1 \to X_2$, with distances $d_1$ and $d_2$ respectively, is an isometry if it is onto, and for all $x, y \in X_1$, $d_1(x, y) = d_2(\varphi(x), \varphi(y))$, also $G \curvearrowright X$ is an *isometric action* if for all $x, y \in X$ and $g \in G$:

$$d(x, y) = d(g \cdot x, g \cdot y).$$

The idea of the metric in the group will be related to the distance between the vertices of the Cayley graph, for this we will introduce the concept of *words* and *paths*.

**Definition 2.1.2.** Given a set $S$, a finite sequence of elements from $S$, possibly with repetition, is called a *word*.

We will construct a free monoid that is going to consist in all possible words of a given set, where the identity is the empty word, and their formal inverses, that is $\{S \cup S^{-1}\}$, and the operation is the concatenation. We remark that this is a monoid, so the element $xx^{-1}$ is not the identity. One last convention that we use is that $(x^{-1})^{-1} = x$. We will denote this monoid as $\{S \cup S^{-1}\}^*$.

Note that if $S$ represents the generating set of a group $G$, then we associate the element $\omega = x_1x_2...x_k \in \{S \cup S^{-1}\}^*$ with an edge path in the Cayley Graph $\Gamma_{G,S}$, where the path starts at the vertex corresponding to the identity, and goes through the graph, as it is dictated by $\omega$.

**Example 2.1.3.** Consider $G = \mathbb{Z} \oplus \mathbb{Z}$, generated by $x = (1, 0)$ and $y = (0, 1)$, and the word $\omega = xxy^{-1}x^{-1}yyyxxx$, the edge path $\mathcal{P}_\omega$ is illustrated in figure 2.1.

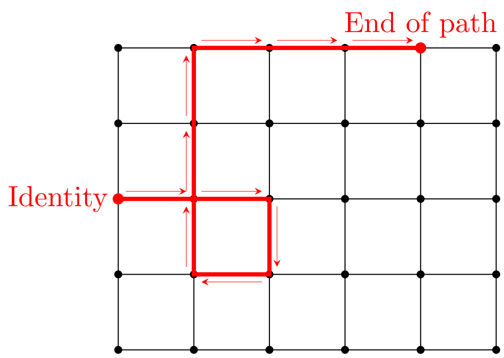



Figure 2.1: Path $\mathcal{P}_\omega$ representing $\omega = xxy^{-1}x^{-1}yyyxxx$

Conversely, every finite path starting at $\{e\}$ in a Cayley graph, describes a word in the generators and their inverses. The product on the group, is the concatenation of the paths. For example if we take $\omega_g = y^2x^3$ and $\omega_h = y^{-3}x$, then $\omega_{g \cdot h} = y^2x^3y^{-3}x$ as is shown in figure 2.2.

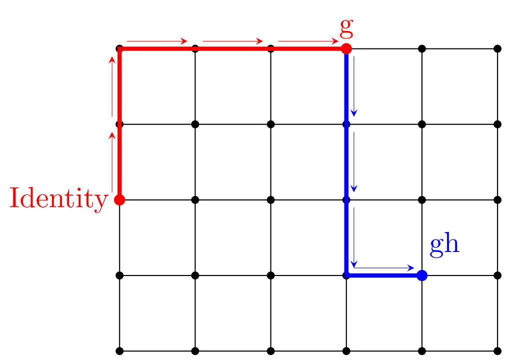



Figure 2.2: Path $\mathcal{P}_{\omega_{gh}}$

The way to connect formally these concepts, is to consider:

$$\pi : \{S \cup S^{-1}\}^* \to G$$

This maps a word in the monoid to the corresponding element of $G$. Since $S$ is a generating set, $\pi$ is onto. So for example, if we let $G = \mathbb{Z} \oplus \mathbb{Z}$, generated by $a = (1, 0)$ and $b = (0, 1)$ $\omega_1 = aba^{-1}bbba$ and $\omega_2 = bbabb$, they are distinct elements on $\{S \cup S^{-1}\}^*$, but $\pi(\omega_1) = \pi(\omega_2) = (1, 4) \in \mathbb{Z} \oplus \mathbb{Z}$. In a similar way, a *normal form* is a function:

$$\eta : G \to \{S \cup S^{-1}\}^*.$$

Such that $\pi \circ \eta : G \to G$ is the identity.

Using this, we can define:

$$d_s(g, h) = \text{the length of the shortest word representing } g^{-1}h.$$

If $\omega$ is a word on $\{S \cup S^{-1}\}^*$, representing $g^{-1}h$, then:

$$g^{-1}h = \pi(\omega) \Rightarrow h = g\pi(\omega)$$

This way, $\omega$ labels a path, connecting the vertices associated to $g$ to the vertex associated to $h$, also note that a minimal-length word, describes a minimal-length path between vertex in the Cayley graph.

**Definition 2.1.4.** The length of $g \in G$ is the amount of generators of the minimal word $\omega \in \{S \cup S^{-1}\}^*$ where $\pi(\omega) = g$. We denote this value as $|g|$.

Is easy to see that in example 2.1.3, if $g = (m, n)$, then $|g| = |m| + |n|$

**Theorem 2.1.5** (Gromov's Corollary)**.** *Every finitely generated group can be faithfully represented as a group of isometries of a metric space.*

*Proof.* The first part of this proof shows that the Cayley graph is a metric space.

Similar to the Cayley's Theorem for groups and Theorem 1.1.5 the metric space is built from the group G, this is because the vertices of the Cayley graph corresponds to elements of $G$ and using the distance $d_s(g, h)$ that was mentioned before. Note that the conditions 1 and 2 of the Definition 2.1.1 are already given, so it only remains to show that the distance function is symmetric and the triangle inequality holds.

If $d_s(g, h) = n$ that means that there exist a word $\omega$ such that $g^{-1}h = \pi(\omega)$, also, $h^{-1}g = \pi(\omega^{-1})$, thus doing one step at a time in the opposite direction, we can see that $d_s(h, g) \leq n$, but if there exists another word $\omega'$ representing $h^{-1}g$, then we could take its formal inverse and form a shorter word that represents $g^{-1}h$ that is a contradiction, therefore our distance is symmetric.

Let $\omega_{gk}$ and $\omega_{kh}$ be the minimal-length words such that $g \cdot \pi(\omega_{gk}) = k$ and $k \cdot \pi(\omega_{kh}) = h$, then $g \cdot \pi(\omega_{gk}\omega_{kh}) = h$, hence

$$d_s(g, h) \leq |\omega_{gk}| + |\omega_{kh}| = d_s(g, k) + d_s(k, h)$$

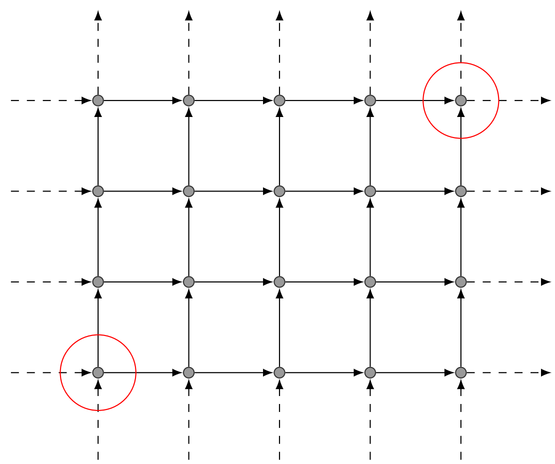
This last property shows that a group $G$ con be viewed as a metric space. Because of that, the natural answer to the question of how the can the group be faithfully represented is given by Cayley's theorem (1.1.4) which shows that left multiplication gives an action of

$G$ on itself, the important thing to notice is that the same representation of $G$ as a group of permutations gives the representation as a group of isometries because that action preserves distances, i.e:

$$d_s(h,k) = \left|h^{-1}k\right| = \left|h^{-1}g^{-1}gk\right| = d_s(gh,gk),$$

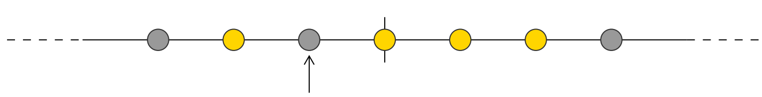
for any $g, h, k \in G$. $\square$

For example, let us consider the next figure that is the same as in 1.4, the distance between the lower left-hand vertex and the upper right-hand vertex is 7. This can be done with $\binom{7}{4}$ different words, four $x'$s and three $y'$s.

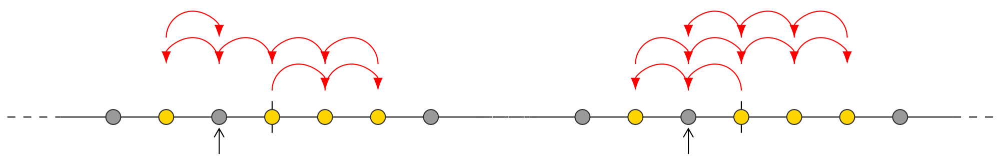


Some natural definitions come after this, the diameter is the minimal integer $D$ such that one can get between any 2 vertex by some edge path of length $\leq D$, a minimal-length path between two vertices is a *geodesic path*, and other definitions.

Finding these geodesic words or diameters is not always as easy as in $\mathbb{Z} \oplus \mathbb{Z}$, the lamplighter group $\mathcal{L}_2$ that was worked in Section 1.2 can give us a more explicit example of the difficulties doing this. Before establishing a general formula for the length of an arbitrary element, consider the element $g \in \mathcal{L}_2$ corresponding to the Figure 1.9 that is a representation of $[\{-2, 0, 1, 2\}, -1]$.



For this, we are going to use the analogy of the lamplighter group given before. If we consider that the lamplighter starts in the zero element, it is evident that the most efficient way to travel lighting the lamps, is either to go first left and then right or vice versa (any other way of traveling implies intermediate steps that make longer the travel of the lamplighter), both represented in 2.3.



(a) Right-first path in $\mathbb{Z}_2 \wr \mathbb{Z}$ (b) Left-first path in $\mathbb{Z}_2 \wr \mathbb{Z}$

Figure 2.3: Different ways of traveling in 1.9

As the lamplighter has to end its travel at the element $\{-1\}$, we can conclude that the most efficient way to do the travel is represented in 2.3b, that is, going to the element that represents the $\{2\}$ lighting each lamp the first time it is passed through, then going to the element of $-2$, (also lighting the lamps when it is necessary), and finally go to $-1$ without lighting that element. Then the geodesic path of this element must involve 11 letters from $\{a, t, t^{-1}\}$, given by: two for the travel to the right, four to the travel to the leftmost vertex, one corresponding to positioning the lamplighter to the vertex numbered by $-1$ and four for each lamp that was lighted up. This gives us a motivation to define these options:

- **The Right-first normal form:** Basically it consists in first move to the rightmost lit lamp, and then move to the leftmost lit lamp, lighting lamps every time it is needed. For the example $g = [\{-2, 0, 1, 2\}, -1] \in \mathcal{L}_2$ corresponding to figure 1.9, we can see that the right-first normal form is:

$$g = atatat^{-4}at.$$

- **The Left-first normal form:** This is identical to the normal form above, only that it moves first to the leftmost lit lamp. Considering again $g = [\{-2, 0, 1, 2\}, -1] \in \mathcal{L}_2$, the left-first normal form is:

$$g = at^{-2}at^3atat^{-3}.$$

Knowing that $|g| = 11$ (because of our previous analysis), it follows that the right-first normal form is a geodesic word, while the left is not. Note that for example in $g = [\{1, 3\}, 2] \in \mathcal{L}_2$ which is shown in Figure 1.10, both forms are the same.

**Proposition 2.1.6.** Let $g = [S, k] \in \mathcal{L}_2$, and let $R = \max\{S \cup 0\}$ and $L = \min\{S \cup 0\}$, then the length of $g$ is given by:

$$|g| = \#(S) + \min\{2R + |L| + |k - L|, 2|L| + R + |k - R|\}.$$

*Proof.* First of all, notice that the elements of $S$ correspond to occurrences of $a$, this is where $\#(S)$ comes from. We can divide the proof in some parts, where we frequently use the analogy of $\mathcal{L}_2$:

**Case 1:** If there are no lit lamps at negative integers, then $L = 0$, then the lamplighter needs to take $R$ steps to the right and then move $|k - R|$ steps to the location of the pointer. As $L = 0$ the formula we gave works.

**Case 2:** Similar to the case above, if there are no lit lamps on positive integers, $R = 0$ and the formula follows.

**Case 3:** If there is a lit lamp at $m < 0$ and at $n > 0$, using the rightmost normal form, there have to be $2R+|L|+|k-L|$ occurrences of $t$ or $t^{-1}$. Similarly using the leftmost normal form there are $2|L| + R + |k - R|$ movements of the lamplighter.

Considering the minimum between the rightmost normal form and the leftmost normal form, the formula follows. □

## 2.2 Thompson's Group $\mathcal{F}$

Given the closed interval $[0, 1]$, we define a *dyadic division* of this, that is constructed fist dividing $[0, 1]$ and $[0, 1/2], [1/2, 1]$, and each sub interval define (or not) another division a finite number of times. For example:

$$[0, 1] = [0, 1/4] \cup [1/4, 1/2] \cup [1/2, 3/4] \cup [3/4, 1]$$

We refer to any interval of the form $\left[\frac{m}{2^n}, \frac{m+1}{2^n}\right]$ $(0 \leq m \leq 2^n - 1)$ as a *standard dyadic interval*, this dyadic intervals can be represented as trees, where the leafs are in correspondence to the divisions of the interval. Using the last example we obtain figure 2.4.

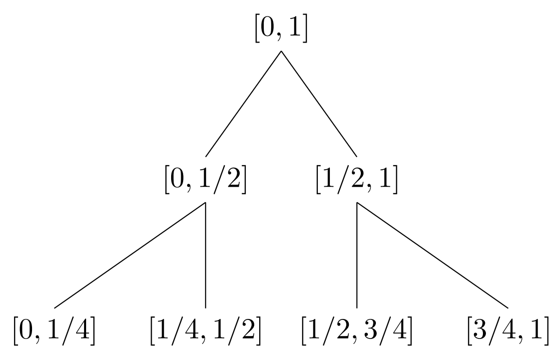


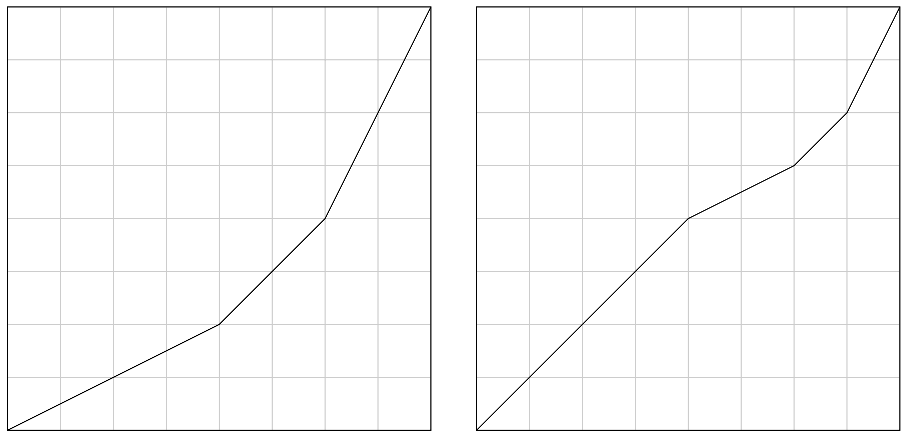

Figure 2.4: Dyadic intervals generate a rooted binary tree

Those are known as "finite, rooted binary trees", in the remainder of this chapter we abbreviate to "frb-trees".

Given an ordered pair of dyadic divisions of $[0, 1]$, with the same number of pieces, there is a corresponding piecewise linear function $f : [0, 1] \to [0, 1]$. Lets denote the chosen middles of the first dyadic division as $0 < m_1 < \cdots < m_k < 1$ and the second ones $0 < \mu_1 < \cdots < \mu_k < 1$, then $f$ is defined as:

1. $f(0) = 0$ and $f(1) = 1$;
2. $f(m_i) = \mu_i$ for all $i$;
3. $f$ is linear restricted to $[0, m_1], [m_i, m_i + 1]$ (for $1 \leq i \leq k$) and $[m_k, 1]$.

**Example 2.2.1.** Consider the 2 dyadic divisions $\zeta_1 = [1, 1/2] \cup [1/2, 3/4] \cup [3/4, 1]$ and $\zeta_2 = [0, 1/4] \cup [1/4, 1/2] \cup [1/2, 1]$, the associated function is the one shown on the figure 2.5a. Another interval example is, is generated by $\xi_1 = \{1/2, 3/4, 7/8\}$ $\xi_2 = \{1/2, 5/8, 3/4\}$ that represents the one shown in 2.5b. We will refer to these elements as *Thompson functions*, and we will show that those are the elements of *Thompson's group* $\mathcal{F}$.



(a) Graph of the Thompson function defined by $\zeta_1$ y $\zeta_2$

(b) Graph of the Thompson function defined by $\xi_1$ y $\xi_2$

Figure 2.5: Two elements of Thompson's Group $\mathcal{F}$

A way to understand this Thompson functions, is thinking of them as elements of the group of homeomorphisms of $([0,1])$ with the composition as their operation. Since dyadic divisions of $[0,1]$ correspond with frb-trees, we can use an ordered pair of frb-trees to describe elements of the group $\mathcal{F}$, we will use $[T_2 \leftarrow T_1]$ to denote the Thompson function where the domain has been divided according to $T_1$ and the range according to $T_2$. Understanding that, the following lemma is given by the function composition.

**Lemma 2.2.2.** Let $T_1, T_2$ and $T_3$ be frb-trees with the same number of leaves, then:

$$[T_3 \leftarrow T_2][T_2 \leftarrow T_1] = [T_3 \leftarrow T_1],$$

and

$$[T_2 \leftarrow T_1]^{-1} = [T_1 \leftarrow T_2].$$

Note that there are many different ordered pairs of frb-trees that represent the same element of $F$, for example the identity is given by $[T \leftarrow T]$ for any frb-tree. If $T$ is a frb-tree, we denote $T \wedge i$ the frb-tree created by adding a wedge to $T$ at the $i$th leaf, an example is figure 2.6, where the original tree is in black and the lighter edges show the result of $(T \wedge 2) \wedge 2$. It is easy to see that $[T_2 \leftarrow T_1]$ and $[T_2 \wedge i \leftarrow T_1 \wedge i]$ are the same functions.

In view of the above we can define the next relation:

$$[T_2 \leftarrow T_1] \sim [T_2 \wedge i \leftarrow T_1 \wedge i]$$

Notice that this is an equivalence relation, the frb-trees are equivalent to pairs with wedges added to the leaves, sometimes pairs of wedges can be deleted, if the leaves enumerated $i$ and $i+1$ form a wedge, this is an *exposed wedge*. If $T_1$ and $T_2$ do not have a pair of matched exposed wedges, then the pair $[T_2 \leftarrow T_1]$ is *reduced*.

**Definition 2.2.3.** The set of Thompson functions forms a group under function composition. It is named the Thompson's group $\mathcal{F}$.

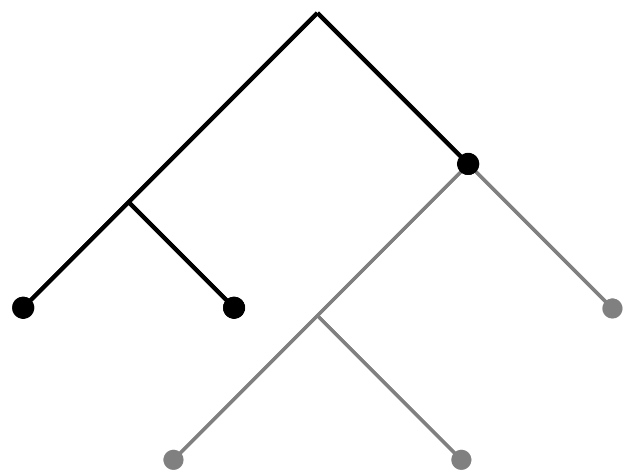


Figure 2.6: $(T \wedge 2) \wedge 2$

A question that is common studying the Thompson's group $\mathcal{F}$ is that if two functions come from frb-trees that have different number of leafs, how can they be operated?. The answer is that because any two dyadic divisions have a common dyadic subdivision where (seen as functions) the domains coincide and therefore they can be composed.

We will denote $\text{Supp}(f) = \{x \in [0,1] : f(x) \neq x\}$ the support of $f$. We call an element $f \in \mathcal{F}$ to be a *left* element if $\text{Supp}(f) \subset (0, 1/2)$, similarly $f$ is a *right* element if $\text{Supp}(f) \subset (1/2, 1)$, the set of left and right elements of $\mathcal{F}$ form subgroups $\mathcal{F}_l$ and $\mathcal{F}_r$ respectively.

Let $l$ to be the homomorphism $\mathcal{F} \to \mathcal{F}$ that takes $f \in \mathcal{F}$ to:

$$f_l(x) = \begin{cases} f(2x)/2 & 0 \leq x \leq 1/2 \\ x & 1/2 \leq x \leq 1 \end{cases}$$

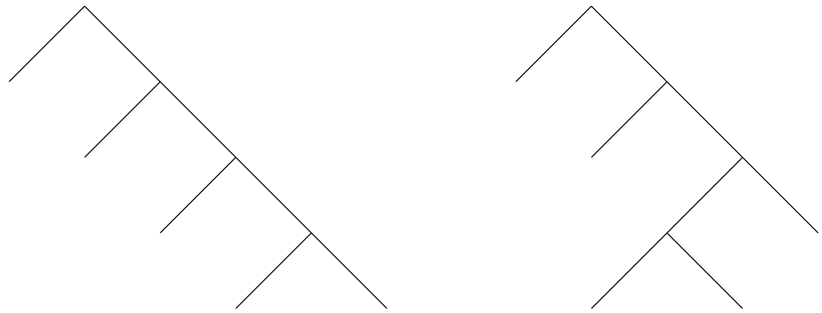
Similarly define $r$ that takes $f \in \mathcal{F}$ to:

$$f_r(x) = \begin{cases} x & 0 \leq x \leq 1/2 \\ 1/2 + f(2x)/2 & 1/2 \leq x \leq 1 \end{cases}$$

The graph of $f_l$ consists of a copy of the graph of $f$ that has been shrunk and tucked into $[0, 1/2] \times [0, 1/2]$ and is extended to the remainder of the domain as the identity. Similarly the graph of $f_r$ is embedded into $[1/2, 1] \times [1/2, 1]$. This homomorphism shows that both $\mathcal{F}_l$ and $\mathcal{F}_r$ are isomorphic to $\mathcal{F}$, since the elements in $\mathcal{F}_l$ and $\mathcal{F}_r$ are disjoint because of the elements of the support, and the elements of both subgroups commute, then $\mathcal{F}_l \times \mathcal{F}_r \cong \mathcal{F} \times \mathcal{F}$, this gives us the proof of the following:

**Proposition 2.2.4.** Thompson's group $\mathcal{F}$ contains a subgroup isomorphic to $\mathcal{F} \times \mathcal{F}$.

In order to understand the Thompson's group $\mathcal{F}$, we want to introduce two families of frb-trees. Lets denote $\mathcal{T}_n$ be the frb-tree where $\mathcal{T}_0$ is a single wedge and $\mathcal{T}_{n+1} = \mathcal{T}_n \wedge (n + 1)$.Let $\mathcal{S}_n$ be the frb-tree where $\mathcal{S}_0 = \mathcal{T}_0$ and $\mathcal{S}_{n+1} = \mathcal{T}_n \wedge n$. Examples of $\mathcal{T}_3$ and $\mathcal{S}_3$ are in 2.7a and 2.7b retrospectively.



(a) $\mathcal{T}_3$ (b) $\mathcal{S}_3$

Figure 2.7: The frb-trees $\mathcal{T}_3$ and $\mathcal{S}_3$

An element $f \in \mathcal{F}$ is said to be *positive* if it corresponds to a pair of the form $[\mathcal{T} \leftarrow \mathcal{T}_n]$; it is negative if it corresponds to a pair $[\mathcal{T}_n \leftarrow \mathcal{T}]$. Since $[\mathcal{T} \leftarrow \mathcal{T}_n]^{-1} = [\mathcal{T}_n \leftarrow \mathcal{T}]$, the negative elements are inverses of positive.

**Lemma 2.2.5.** Every element $f \in \mathcal{F}$ can be expressed as a product of a positive and a negative element.

*Proof.* Let $f$ be a Thompson function given by $[S \leftarrow T]$ where $S$ and $T$ have $n+1$ leaves, then

$$f = [S \leftarrow T] = [S \leftarrow \mathcal{T}_n][\mathcal{T}_n \leftarrow T].$$

Because of Lemma 2.2.2, then f is expressed as the product of a positive and a negative element.

□

Let us define $x_i$, the Thompson function described by $[\mathcal{S}_{n+1} \leftarrow \mathcal{T}_{n+1}]$. Note that the graphs of $x_0$ and $x_1$ are shown in Figure 2.5; the collection of $x_i$ satisfies the following relations:

**Lemma 2.2.6.** If $i < n$ then ${x_i}^{-1} x_n x_i = x_{n+1}$.

*Proof.* First of all, notice that because $i < n$ then $\mathcal{T}_n \wedge n \wedge i = \mathcal{T}_n \wedge i \wedge (n+1)$, and our objective is to show that:

$$[\mathcal{T}_{i+1} \leftarrow \mathcal{S}_{i+1}][\mathcal{S}_{n+1} \leftarrow \mathcal{T}_{n+1}][\mathcal{S}_{i+1} \rightarrow \mathcal{T}_{i+1}] = [\mathcal{S}_{n+2} \leftarrow \mathcal{T}_{n+2}],$$

notice that $[\mathcal{S}_{i+1} \leftarrow \mathcal{T}_{i+1}] = [\mathcal{T}_{n+1} \wedge i \leftarrow \mathcal{T}_{n+2}]$, then

$$[\mathcal{T}_{i+1} \leftarrow \mathcal{S}_{i+1}][\mathcal{S}_{n+1} \leftarrow \mathcal{T}_{n+1}][\mathcal{S}_{i+1} \rightarrow \mathcal{T}_{i+1}] = [\mathcal{T}_{i+1} \leftarrow \mathcal{S}_{i+1}][\mathcal{S}_{n+1} \wedge i \leftarrow \mathcal{T}_{n+2}].$$

Similarly

$$[\mathcal{T}_{i+1} \leftarrow \mathcal{S}_{i+1}][\mathcal{S}_{n+1} \wedge i \leftarrow \mathcal{T}_{n+2}] = [\mathcal{T}_{n+1} \wedge (n+1) \leftarrow \mathcal{T}_n \wedge i \wedge (n+1)][\mathcal{T}_n \wedge n \wedge i \leftarrow \mathcal{T}_{n+2}]$$
$$= [\mathcal{S}_{n+2} \leftarrow \mathcal{T}_{n+2}]$$

□

**Lemma 2.2.7.** If $i < n+2$ then $[T \leftarrow \mathcal{T}_n] \cdot x_i = [T \wedge i \leftarrow \mathcal{T}_{n+1}]$.

*Proof.* By definition, $x_i = [\mathcal{S}_{i+1} \leftarrow \mathcal{T}_{i+1}]$ and since $i < n+2$, $\mathcal{T}_n \cup \mathcal{S}_{i+1} = \mathcal{T}_n \wedge i$, thus:

$$\begin{aligned}
[T \leftarrow \mathcal{T}_n] \cdot x_i &= [T \leftarrow \mathcal{T}_n] \cdot [\mathcal{S}_{i+1} \leftarrow \mathcal{T}_{i+1}] \\
&= [T \leftarrow \mathcal{T}_n] \cdot [\mathcal{T}_n \wedge i \leftarrow \mathcal{T}_{n+1}] \\
&= [T \wedge i \leftarrow \mathcal{T}_n \wedge i] \cdot [\mathcal{T}_n \wedge i \leftarrow \mathcal{T}_{n+1}] \\
&= [T \wedge i \leftarrow \mathcal{T}_{n+1}].
\end{aligned}$$

□

**Theorem 2.2.8.** *$\mathcal{F}$ is generated by the set of positive elements $\{x_0, x_1, x_2, \dots\}$.*

*Proof.* Note that it suffices to show that every positive element of $\mathcal{F}$ is a product of the $x_i$'s. For this, let $[T \leftarrow \mathcal{T}_n]$ and note that there is a maximal sub-tree of the form $\mathcal{T}_k$ such that $T = \mathcal{T}_k \wedge i_1 \wedge i_2 \wedge \cdots \wedge i_m$, then Lemma 2.2.7 shows that:

$$[T \leftarrow \mathcal{T}_n] = [\mathcal{T}_k \leftarrow \mathcal{T}_k] \cdot x_{i1} \cdot x_{i2} \cdot \dots \cdot x_{im},$$

so

$$[T \leftarrow \mathcal{T}_n] = x_{i1} \cdot x_{i2} \cdot \dots \cdot x_{im}.$$

□

**Corollary 2.2.9.** Thompson's group $\mathcal{F}$ is generated by $x_0$ and $x_1$.

*Proof.* The Lemma 2.2.6 shows that $x_2 = {x_0}^{-1} x_1 x_0$, and an induction argument shows that $x_{n+1} = {x_0}^{-n} x_1 {x_0}^n$. The result follows by the theorem above.

□

**Theorem 2.2.10.** *The following is a presentation for Thompson's group $\mathcal{F}$.*

$$\langle x_0, x_1, x_2, \dots \,|\, x_k^{-1} x_n x_k = x_{n+1} \textit{ for } k < n \rangle.$$

The proof of this fact is long and can be found in [5]. And the following theorem can be found as Theorem 4.8 of [4].

**Theorem 2.2.11.** *Every non-abelian subgroup of $\mathcal{F}$ contains a free abelian subgroup of infinite rank*

A well known fact of free groups is that every subgroup of a free group is itself free (known as Nielsen-Schreier theorem, it can be found as theorem 1A.4. of [9]). This and the above theorem implies that $\mathcal{F}$ does not contain a subgroup isomorphic to $\mathbb{F}_2$, this is going to be a motivation to consider this group as a candidate for a group with intermediate growth as we see in the following section.

## 2.3 The Growth of Groups

In the Section 2.1 we give a notion of geometry of a group, this will be use to define the growth of a group. First we define any non-decreasing function $f : [0, \infty) \to [0, \infty)$ to be a growth function. On the other hand let $G$ be a group with finite generating set $S$, and let $\mathcal{B}_S(e, n)$ denote the ball of radius $n$ about the identity in $G$, i.e

$$\mathcal{B}_S(e, n) = \{h \in G : d_s(e, h) \leq n\}.$$

In a similar way $\mathcal{S}(e, n)$ the sphere of radious $n$ as:

$$\mathcal{S}(g, n) = \{h \in G : d_s(g, h) = n\}.$$

**Definition 2.3.1.** The *Spherical growth function* of a group $G$ with respect to a generating set $S$ is the function $\sigma : \mathbb{N} \to \mathbb{N}$ defined as $\sigma(n) = |\mathcal{S}(e, n)|$, where $|\mathcal{S}(e, n)|$ is the amount of elements in $\mathcal{S}(e, n)$. The associated growth series is the formal power series:

$$\mathcal{S}(z) = \sum_{n \geq 0} \sigma(n) z^n.$$

**Example 2.3.2.** Let $G = \mathbb{Z}$ with a single generator. Then

$$\sigma(n) = \begin{cases} 1 & n = 0, \\ 2 & n > 0. \end{cases}$$

and so the associated growth series is $\mathcal{S}(z) = 1 + 2z + 2z^2 + 2z^3 + \dots$, and it can be expresed as:

$$\mathcal{S}(z) = \frac{1+z}{1-z}$$

It is important to notice that this series depends on the generating set, for example if $G = \mathbb{Z}$ but generated by $\{2, 3\}$, then the associated growth function will be:

$$\sigma(n) = \begin{cases} 1 & n = 0 \\ 4 & n = 1 \\ 8 & n = 2 \\ 6 & n \geq 3 \end{cases}$$

And so the series is:

$$\mathcal{S}(z) = \frac{1 + 3z + 4z^2 - 2z^3}{1-z}$$

There are important properties about this series, for example:

**Theorem 2.3.3.** *Let $G$ and $H$ be groups with finite generating sets $\mathcal{S}_G$ and $\mathcal{S}_H$, and corresponding growth series $\mathcal{S}_G(z)$ and $\mathcal{S}_H(z)$. Then*

$$\mathcal{S}_{G \oplus H} = \{(s, e_h) : s \in \mathcal{S}_G\} \cup \{(e_g, h) : h \in \mathcal{S}_H\}$$

*is a generating set for $G \oplus H$ and the corresponding growth series is given by*

$$\mathcal{S}_{G \oplus H}(z) = \mathcal{S}_G(z) \cdot \mathcal{S}_H(z)$$

*Proof.* The claim about the generating set is clear. And the length of $(g, h) \in G \oplus H$ is the sum of the lengths of $g$ and $h$, so if we want an element to have length $n$, it can be done in all the different forms that the sum of elements of $G$ and elements of $H$ is exactly $n$, that is:

$$\sigma_{G\oplus H}(n) = \sum_{i=0}^{n} \sigma_G(i) \cdot \sigma_H(n-i),$$

from which the statement follows. □

**Corollary 2.3.4.** The growth series from $\mathbb{Z}^n = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\text{n copies}}$ with respect to a standard generating set is

$$\mathcal{S}_{\mathbb{Z}^n}(z) = \left(\frac{1+z}{1-z}\right)^n.$$

There are many other properties, for example not all growth series of finitely generated groups are rational, for example the growth series of $\mathbb{Z}_2 \wr \mathbb{Z}$ is not rational (see [2] for details on this).

All we have done depends on the generating set of the group, and some properties of the graphs are lost in some sets, so we want to consider some special properties, we are going to refer to these as *large-scale properties*. A property of a Cayley graph of a finitely generated group is large-scale only if it is invariant under changes in the generating set. For example, using this convention, having a rational growth series is not a large-scale property (see [3]).

**Definition 2.3.5.** Let $\Gamma$ and $\Lambda$ be two graphs, a *map* from $\Gamma$ to $\Lambda$ is a function $\phi$ taking vertices of $\Gamma$ to vertices of $\Lambda$, and edges of $\Gamma$ to edges of $\Lambda$, such that if $v$ and $w$ are vertices attached to an edge $l \in \Gamma$ then $\phi(l)$ joins $\phi(v)$ to $\phi(w)$.

**Proposition 2.3.6.** Let $S$ and $T$ be two finite generating sets for a group $G$ and let $\Gamma_S$ and $\Gamma_T$ the corresponding Cayley graphs. Then there are maps $\phi_{T\leftarrow S} : \Gamma_S \to \Gamma_t$ and $\phi_{S\leftarrow T} : \Gamma_T \to \Gamma_S$ such that:

1. The compositions $\phi_{T\leftarrow S} \circ \phi_{S\leftarrow T}$ and $\phi_{S\leftarrow T} \circ \phi_{T\leftarrow S}$ induce the identity on $V(\Gamma_S)$ and $V(\Gamma_T)$ respectively.

2. There is a constant $K > 0$ such that the image of any edge $l \in \Gamma_S$ under $\phi_{S\leftarrow T} \circ \phi_{T\leftarrow S}$ is contained in the ball $\mathcal{B}(v, K) \subset \Gamma_S$ where $v$ is a vertex that is joined to $l$. A similar statement also holds for edges of $\Gamma_T$.

*Proof.* If $v_g$ denotes the vertex corresponding to $g$ in $\Gamma_S$ and $v_g{}'$ the vertex in $\Gamma_T$, then $\phi_{T\leftarrow S}(v_g) = v_g{}'$, in a similar way it is defined $\phi_{S\leftarrow T}$, and the first claim follows.

For each generator $s \in S$ we can choose a word $\omega_s = t_1 t_2 \cdots t_k \in \{T \cup T^{-1}\}^*$ such that $s = \pi(\omega_s) \in G$. By the construction of the Cayley graph, if $e$ is an edge of $\Gamma_S$, then $l$ is labeled by a generator $s \in S$ and joins the vertex associated to $g$ to the vertex associated to $g \cdot s$. Then the map $\phi_{T\leftarrow S}$ sends such edge to the edge path

$$g \longrightarrow g \cdot t_1 \longrightarrow gt_1 \cdot t_2 \longrightarrow \cdots \longrightarrow gt_1 t_2 \cdots t_k$$

in $\Gamma_T$, and $\phi_{S\leftarrow T}$ has a similar behavior with a word $\omega_t \in \{S \cup S^{-1}\}^*$. Let $k$ be the maximal length of the words $\omega_t$ and $\omega_s$, it follows that $\phi_{T\leftarrow S}(l)$ is an edge path of length $\leq k$ and the image $\phi_{S\leftarrow T}$ of this path is then an edge path of length $\leq k^2$, then the constant $K$ in the second claim can be taken to be $k^2$. □

**Corollary 2.3.7.** Let $G$, $S$ and $T$ as above, then there is a constant $\lambda \geq 1$ such that for any $g$ and $h$ in $G$,

$$\frac{1}{\lambda} d_S(g,h) \leq d_T(g,h) \leq \lambda d_S(g,h).$$

*Proof.* Let

$$\Lambda_1 = \max\{|\omega_t| : t \in T\}$$

be the maximum length of the words in $\{S \cup S^{-1}\}^*$ which were chosen to represent the generators in $T$. If $d_T(g,h) = n$ by definition there is an edge path between $v_g$ and $v_h$ in $\Gamma_T$ of length $n$. This map under $\phi_{S\leftarrow T}$ has length at most $\Lambda_1 \cdot n$, thus $d_S(g,h) \leq \Lambda_1 \cdot d_T(g,h)$. Repeating this argument with $S$ and $T$ reversed, taking $\Lambda_2 = \mathrm{Max}\{|\omega_s| : s \in S\}$, then $d_T(g,h) \leq \Lambda_2 \cdot d_S(g,h)$, if we set $\lambda = \mathrm{Max}(\Lambda_1, \Lambda_2)$, then the stated inequality holds. □

**Theorem 2.3.8.** *Let $S$, $T$ and $G$ as above lets define the function $\beta_S(n) = |\mathcal{B}_S(e,n)|$ that gives the number of elements of $G$ inside the ball of radius $n$, so there is a constant $\lambda \geq 1$ such that*

$$\beta_S\left(\frac{1}{\lambda}n\right) \leq \beta_T(n) \leq \beta_S(\lambda n)$$

*for all $n \in \mathbb{N}$.*

*Proof.* Let $|g|_S$ denote the length of $g \in G$ with respect to the generating set $S$ and similarly $|g|_T$, the last corollary (2.3.7) give us a $\lambda \geq 1$ such that $|g|_T \leq \lambda \cdot |g|_S$ thus if $g \in \mathcal{B}_T(n)$ then $g \in \mathcal{B}_S(\lambda n)$ then $\beta_T(n) \leq \beta_S(\lambda n)$. Exchanging the roles of $S$ and $T$ establish the other inequality. □

This last theorem shows an interesting behavior of the growth functions of the groups, and it is a motivation for the next definition.

**Definition 2.3.9.** Define $\preceq$ to be the relation on growth functions defined by $f \preceq g$ if there is a constant $\lambda \geq 0$ such that

$$f(x) \leq \lambda g(\lambda x + \lambda) + \lambda$$

for all $x \in [0, \infty)$. We say that if $f \preceq g$ then $g$ dominates $f$. This definition is the result of pre- and post-composing $g(x)$ with the linear function $y = \lambda x + \lambda$. If $f \preceq g$ and $g \not\preceq f$ then $g$ strictly dominates $f$ denoted by $f \prec g$. And if $f \preceq g$ and $g \preceq f$ then $f$ and $g$ are said to be equivalent, denoted by $f \sim g$. Two equivalent growth functions are said to grow at the same rate. It is easy to prove that "$\preceq$" is a reflexive and transitive relation, and "$\sim$" is an equivalence relation.

Note that Theorem 2.3.8 implies that if $S$ and $T$ are two finite generating sets for a group $G$, then the associated growth functions are equivalent, i.e, $\beta_S(n) \sim \beta_T(n)$. Then the equivalent class of a growth function for a group is a large-scale invariant of the group.

**Example 2.3.10.** Consider the free group of rank $k$, along with a fixed basis, it is easy to show that the number of elements in the sphere of radius $\mathcal{S}(n)$ is $2k\cdot(2k-1)^{n-1}$, notice that for each $n$ there are $(2k-1)$ possible elements added for each of the initials $2k$ generators for $n \geq 1$, therefore

$$(2k-1)^n < |\mathcal{S}(n)| < |\mathcal{B}(n)|\,.$$

Hence $(2k-1)^n \preceq \beta(n)$. On the other hand,

$$\begin{aligned}\beta(n) = \sum_{i=0}^{n} |\mathcal{S}(i)| &= 1 + 2k + 2k(2k-1) + \cdots + 2k(2k-1)^{n-1} \\ &< 1 + 2k + (2k)^2 + \cdots + (2k)^{n-1} < (2k)^n\end{aligned}$$

So $\beta(n) \preceq (2k)^n$. The next lemma will help us to finish the example showing that these bounding functions are equivalent.

**Lemma 2.3.11.** Let $a$ and $b$ be two integers greater than 1, and $\alpha(n) = a^n$, $\beta(n) = b^n$ be the corresponding functions, then $\alpha(n) \sim \beta(n)$.

*Proof.* With out loss of generality, assume that $a < b$ thus $\alpha(n) \preceq \beta(n)$. Conversely, if $\lambda \geq \log_a(n)$, then $\beta(n) \leq \alpha(\lambda n)$, so $\beta(n) \preceq \alpha(n)$ . □

With this, we can notice that in the definition of exponential growth, we can change the base 2 with any other integer $\geq 2$, so finally we can say that every finitely generated free group whose rank is at least 2, has exponential growth.

And interesting fact that can be derived from this example, is that every finitely generated group $G$ has its growth dominated by $2^n$, noticing that if $G$ has a generator set $S$ consisting of $k$ elements, their growth function is bounded by the growth function of $\mathbb{F}_k$. That observation gives us the next theorem:

**Theorem 2.3.12.** *If $G$ is a finitely generated group, its growth is dominated by $2^n$.*

In 1968 Jhon Milnor asked if there are groups of "intermediate growth", i.e groups whose growth function strictly dominates $n^d$ for all $d$ but is not equivalent to $2^n$. As we mentioned before, Thompson's group $\mathcal{F}$ contains a copy of $\mathcal{F} \times \mathcal{F}$ (Proposition 2.2.4), but also $\mathcal{F}$ does not contain a non-abelian free group (Theorem 2.2.11), so perhaps the growth of $F$ is not exponential. The next theorem give us an answer to the growth of Thompson's group $\mathcal{F}$.

**Theorem 2.3.13.** *Thompson's group $\mathcal{F}$ has exponential growth.*

*Proof.* Our objective is to show that there is no element in $\mathcal{F}$ that can be represented as two different words in $\{x_0, x_1\}^*$. Something really important to notice is that we are ignoring the inverses. If we can prove that claim, it will tell us that there are at least $2^n$ words of length $n$ in $\{x_0, x_1\}^*$, then $2^n \leq \beta(n)$.

Let us recall that Thompson's group $\mathcal{F}$ is generated by $x_0$ and $x_1$ (Theorem 2.2.9), consider $\omega_0 \neq \omega_2$ and the evaluation map $\pi$ that was defined in Section 2.1. Via contradiction, let us suppose that $\pi(\omega_0) = \pi(\omega_2)$, choosing the combined length $(|\omega_0| + |\omega_2|)$ as short as possible. Note that the last letters of both words must be different, if they are not, we remove the last letter from both words forming $\omega_0'$ and $\omega_1'$, but

$$\begin{aligned}
\pi\left(\omega_0'\right) &= \pi\left(\omega_0 \cdot x^{-1}\right) \\
&= \pi\left(\omega_0\right)\pi\left(x^{-1}\right) \\
&= \pi\left(\omega_1\right)\pi\left(x^{-1}\right) \\
&= \pi\left(\omega_1'\right),
\end{aligned}$$

and their combined length has been reduced, which is not possible.

Without loss of generality assume that $\omega_0$ ends in $x_0$ and $\omega_1$ in $x_1$. Consider an arbitrary element $f \in \mathcal{F}$, when it is restricted to a small interval $[0, \varepsilon)$, $f$ is a linear function with slope $2^k$ for some $k \in \mathbb{Z}$. The function $\varphi : \mathcal{F} \to \mathbb{Z}$ that takes $f$ to the exponent $k$ is a homomorphism, and in particular $\varphi(x_0) = -1$ and $\varphi(x_1) = 0$. Since both words represent the same element, $\varphi(\omega_0) = \varphi(\omega_1)$.

It is important to notice that $|\varphi(\omega_0)| = |\varphi(\omega_1)|$ is the number of $x_0$'s in the words (because the slope of $x_1$ and using that we are ignoring the inverses), call that number $\mathfrak{n}$. Note that $\mathfrak{n} > 0$; otherwise both words would be powers of $x_1$.
Think of $x_0$ as the function, $x_0(3/4) = 1/2$, and more generally $x_0^k(3/4) = 1/2^k$ for $k \in \mathbb{N}$. Further, since $x_1$ is the identity when it is restricted to $[0, 1/2]$, and $\omega_0$ ends in $x_0$, then $\pi(\omega_0)$ takes $3/4$ to $1/2^{\mathfrak{n}}$ (the same $\mathfrak{n}$ above).

On the other hand consider now the action of $\pi(\omega_1)$ on $[0, 1]$. There is some positive integer $m$ such that $\omega_1$ ends in $x_0 x_1^m$, note that $x_1$ takes $3/4$ to a number that is smaller than $3/4$, and therefore $x_0 x_1^m$ takes $3/4$ to a number that is strictly smaller than $1/2$. So it is impossible that both words represent the same element. It follows that $\mathcal{F}$ has exponential growth. □

Sadly Thompson's group $\mathcal{F}$ does not answer Milnor's question. In 1983, Grigorchuck proved the following:

**Theorem 2.3.14.** *There are finitely generated groups $G$ with growth function $\beta$ where*

$$n^d \prec \beta \prec 2^n$$

*for all $d$.*

# Chapter 3

# Quasi-isometries

In the second chapter we introduce the idea of large-scale properties in groups, an interesting property in the comparison of metric spaces is the existence of quasi-isometries they can see the coarse structure of a metric space ignoring the local structure.
In this chapter we define the quasi-isometries and gave a direct application of this concept to the relation of groups and metric spaces,by means of the Švarc-Milnor Lemma, that we prove. Finally we end up with some quasi-isometric invariants and properties that are very interesting to study.

## 3.1 Quasi-isometries

We already defined a metric space in Definition 2.1.1, with this in mind, we introduce the following concepts:

**Definition 3.1.1.** Let $f\colon X \longrightarrow Y$ a function between metric spaces $(X, d_X)$ and $(Y, d_Y)$, we say that $f$ is an isometric embedding if

$$\forall x, x' \in X \quad d_Y\left(f(x), f\left(x'\right)\right) = d_X\left(x, x'\right).$$

The map is an isometry if there is an inverse isometric embedding such that the composition of those are the respective identities.

The notion of isometry between spaces is too rigid for our purposes, it preserves all local details, and we are looking for a condition that represents the large scales properties. That leads us to give the following definition:

**Definition 3.1.2.** Let $f\colon X \longrightarrow Y$ be a map between metric spaces $(X, d_X)$ and $(Y, d_Y)$.

- The map $f$ is a $(\delta, \varepsilon)-$quasi-isometric embedding (or simply quasi-isometric embedding) if there are constants $\delta \in \mathbb{R}_{\geq 1}, \varepsilon \in \mathbb{R}_{\geq 0}$ such that:

$$\forall x, x' \in X \quad \frac{1}{\delta} \cdot d_X\left(x, x'\right) - \varepsilon \leq d_Y\left(f(x), f\left(x'\right)\right) \leq \delta \cdot d_X\left(x, x'\right) + \varepsilon$$

- A map $g : X \longrightarrow Y$ has finite distance from $f$ if there is a constant $\kappa \in \mathbb{R}_{\geq 0}$ with:

$$\forall x \in X : \quad d_X(f(x), g(x)) \leq \kappa.$$

- The map $f$ is a quasi-isometry if it is a quasi-isometric embedding for which there is another quasi-isometric embedding $g : X \longrightarrow Y$ such that $g \circ f$ has finite distance from $\mathrm{id}_X$ and $f \circ g$ has finite distance from $\mathrm{id}_Y$.

If there is a quasi-isometry between $X$ and $Y$ we say that the two metric spaces are quasi-isometric, and we denote it as $X \sim_{\mathrm{QI}} Y$. In the literature equivalent definitions of this concept can be found, but the most common is the one that we are using.

**Theorem 3.1.3.** *The property of quasi-isometry is an equivalence relation.*

*Proof.* The reflexive and symmetric properties follow by definition. For the transitive property, consider functions:

$$f\colon X \longrightarrow Y, \quad \tilde{f}\colon Y \longrightarrow Z,$$
$$g\colon Y \longrightarrow X, \quad \tilde{g}\colon Z \longrightarrow Y.$$

where all of them are quasi-isometric embeddings, and there exists $\kappa_1, \kappa_2, \tilde{\kappa}_1, \tilde{\kappa}_2 \in \mathbb{R}_{\geq 0}$ such that:

$$d_X((g \circ f)(x), \mathrm{id}_X(x)) \leq \kappa_1, \quad d_Y((\tilde{g} \circ \tilde{f})(y), \mathrm{id}_Y(y)) \leq \tilde{\kappa}_1,$$
$$d_Y((f \circ g)(y), \mathrm{id}_Y(y)) \leq \kappa_2, \quad d_Z((\tilde{f} \circ \tilde{g})(z), \mathrm{id}_Z(z)) \leq \tilde{\kappa}_2.$$

Since $f$ and $\tilde{f}$ are quasi-isometric embeddings, we have that:

$$\frac{1}{\delta_1} d_X(x_1, x_2) - \varepsilon_1 \leq d_Y(f(x_1), f(x_2)) \leq \delta_1 d_X(x_1, x_2) + \varepsilon_1,$$
$$\frac{1}{\tilde{\delta_1}} d_Y(y_1, y_2) - \tilde{\varepsilon_1} \leq d_Z(\tilde{f}(y_1), \tilde{f}(y_2)) \leq \tilde{\delta_1} d_Y(y_1, y_2) + \tilde{\varepsilon_1}.$$

Note that we can consider $y_1 = f(x_1)$, then using both inequalities we get:

$$\frac{1}{\delta_1 \tilde{\delta_1}} d_X(x_1, x_2) - (\varepsilon_1 \tilde{\delta_1} + \tilde{\varepsilon_1}) \leq d_Z((\tilde{f} \circ f)(x_1), (\tilde{f} \circ f)(x_2)) \leq \delta \tilde{\delta_1} d_X(x_1, x_2) + (\varepsilon_1 \tilde{\delta_1} + \tilde{\varepsilon_1})$$

This shows that $(\tilde{f} \circ f)$ is a quasi-isometric embedding. In a similar way we can prove that $(g \circ \tilde{g})$ is also a quasi-isometric embedding.

Also, note that

$$d_Z((\tilde{f} \circ f) \circ (g \circ \tilde{g})(z), \mathrm{id}_Z(z)) \leq \kappa_2 + \tilde{\kappa}_2,$$
$$d_X((g \circ \tilde{g}) \circ (\tilde{f} \circ f)(x), \mathrm{id}_X(x)) \leq \kappa_1 + \tilde{\kappa}_1.$$

Then we can conclude that $X$ and $Z$ are quasi-isometric. □

**Example 3.1.4.** Let us define the diameter of an space $X$ as:

$$D = \operatorname{diam} X := \sup_{x,y \in X} d_X(x, y).$$

Any metric space with finite diameter is quasi-isometric to a point. Let us consider the constant function

$$f : X \longrightarrow \bullet,$$

that maps any element of $X$ to the point $\bullet$ , using $\varepsilon = \mathrm{diam}(X)$, we get

$$d_X(x, y) - D \leq d_\bullet(f(x), f(y)) \leq d_X(x, y) + D,$$

the quasi-isometric embedding from the point to $X$ is trivial, and as the diameter is finite, the condition of finite distance is clear.

We can give an alternative characterization of the quasi isometries. For this, consider the following definition:

**Definition 3.1.5.** A map $f : X \to Y$ between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is said to have quasi dense image if there is a constant $c \in \mathbb{R}$ such that:

$$\forall y \in Y, \quad \exists x \in X : \quad d_Y(f(x), y) \leq c.$$

**Proposition 3.1.6.** A map $f : X \to Y$ between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is a quasi-isometry if and only if it is a quasi-isometric embedding with quasi-dense image.

*Proof.* First, if $f : X \to Y$ is a quasi-isometry, by definition there exists a quasi-inverse quasi-isometric embedding $g : Y \to X$ and also the composition $f \circ g$ has finite distance from $\mathrm{id}_Y$, that is:

$$\forall y \in Y \quad d_Y(f \circ g(y), y) \leq c,$$

for some $c \in \mathbb{R}$, note that using $g(y) = x$, we can say that $f$ has quasi-dense image.

Conversely, suppose that $f : X \to Y$ is a quasi-isometric embedding with quasi-dense image. We will construct a quasi-inverse quasi isometric embedding.

By definition, $f$ is a quasi-isometric embedding and from the fact that it has quasi-dense image, there exists a $c \in \mathbb{R}$ such that:

$$\forall x, x' \in X \quad \frac{1}{c} \cdot d_X\left(x, x'\right) - c \leq d_Y\left(f(x), f\left(x'\right)\right) \leq c \cdot d_X\left(x, x'\right) + c,$$

Also $\forall y \in Y, \quad \exists x \in X : \quad d_Y(f(x), y) \leq c$ and we can define (by the axiom of choice) a map:

$$\begin{aligned} g : Y &\longrightarrow X \\ y &\longmapsto x_y, \end{aligned}$$

such that $d_Y(f(x_y), y) \leq c$ for all $y \in Y$. By construction for all $y \in Y$

$$d_Y(f \circ g(y), y) = d_Y\left(f\left(x_y\right), y\right) \leq c,$$

and conversely because of the fact that $f$ is a quasi-isometric embedding, for all $x \in X$ we obtain:

$$d_X(g \circ f(x), x) = d_X\left(x_{f(x)}, x\right) \leq c \cdot d_Y\left(f\left(x_{f(x)}\right), f(x)\right) + c^2 \leq c \cdot c + c^2 = 2 \cdot c^2.$$

Therefore $g$ is a quasi-inverse to $f$. Let $y, y' \in Y$, then using the triangle inequality we get:

$$\begin{aligned} d_X\left(g(y), g\left(y'\right)\right) &= d_X\left(x_y, x_{y'}\right) \\ &\leq c \cdot d_Y\left(f\left(x_y\right), f\left(x_{y'}\right)\right) + c^2 \\ &\leq c \cdot \left(d_Y\left(f\left(x_y\right), y\right) + d_Y\left(y, y'\right) + d_Y\left(f\left(x_{y'}, y'\right)\right)\right) + c^2 \\ &\leq c \cdot \left(d_Y\left(y, y'\right) + 2 \cdot c\right) + c^2 \\ &= c \cdot d_Y\left(y, y'\right) + 3 \cdot c^2, \end{aligned}$$

and note that:

$$d(y, y') \leq d(y, f(x_y)) + d(f(x_y), d(x_{y'})) + d(f(x_{y'}), y'),$$

so

$$d(f(x_y), d(x_{y'}) \geq d(y, y')) - d(y, f(x_y)) - d(f(x_{y'}), y').$$

Using this we get:

$$\begin{aligned} d_X\left(g(y), g\left(y'\right)\right) &= d_X\left(x_y, x_{y'}\right) \\ &\geq \frac{1}{c} \cdot d_Y\left(f\left(x_y\right), f\left(x_{y'}\right)\right) - 1 \\ &\geq \frac{1}{c} \cdot \left(d_Y\left(y, y'\right) - d_Y\left(f\left(x_y\right), y\right) - d_Y\left(f\left(x_{y'}\right), y'\right)\right) - 1 \\ &\geq \frac{1}{c} \cdot d_Y\left(y, y'\right) - \frac{2 \cdot c}{c} - 1 \\ &= \frac{1}{c} \cdot d_Y\left(y, y'\right) - 3, \end{aligned}$$

Taking both inequalities and using $d = \max\left\{3 \cdot c^2, 3\right\}$ we get:

$$\frac{1}{c} \cdot d_Y\left(y, y'\right) - d \leq d_X(g(y), g(g')) \leq c \cdot d_Y\left(y, y'\right) + d.$$

□

**Example 3.1.7.** For $n \in \mathbb{N}$, $\mathbb{Z}^n$ is quasi-isometric to the euclidean space $\mathbb{R}^n$. Note that the natural inclusion $\iota : \mathbb{Z}^n \longrightarrow \mathbb{R}^n$ is a quasi-isometric embedding with quasi-dense image.

**Example 3.1.8.** $\mathbb{R}^n \nsim_{\mathrm{QI}} \mathbb{R}^m$ for $n \neq m$.

In general the quasi-isometries are not continuous, so we will construct a continuous map approximating an hypothetical function that will violate Borsuk-Ulam theorem.

The Borsuk-Ulam theorem says that if $f : S^n \longrightarrow \mathbb{R}^n$ is continuous then there exist an $x \in S^n$ such that $f(-x) = f(x)$.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a $(\lambda, K)$-quasi-isometry and $n > m$. First of all, let us construct a continuous map approximating $f$. Consider the cube grid in $\mathbb{R}^n$ with vertices at $\mathbb{Z}^n$, we can subdivide each $n$-cube into $n$-simplices that give a triangulation of $\mathbb{R}^n$. Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a map which agrees with $f$ on the integer coordinates, and elsewhere is given by a linear interpolation with respect to the triangulation.

Notice that if $x \in \mathbb{R}^n$, there exists $y \in \mathbb{Z}^n$ with $d(x, y) \leq \sqrt{n}/2$. Let $z \in \mathbb{Z}^n$ be a point in the $n$-simplex that contains $x$ that is furthest from $y$, we can use the fact that $f$ is a quasi-isometry and the triangular inequality to show that:

$$\begin{aligned} d(f(x), g(x)) &\leq d(f(x), f(y)) + d(f(y), g(y)) + d(g(y), g(x)) \\ &\leq (\lambda\sqrt{n}/2 + K) + 0 + (\lambda d(y, z) + K) \\ &\leq \frac{3}{2}\lambda\sqrt{n} + 2K. \end{aligned}$$

Consider the inclusion $\iota : S^m \longrightarrow \mathbb{R}^n$, which embeds $S^m$ as a sphere of radius R into $\mathbb{R}^n$, notice that $g \circ \iota$ is a continuous map.

If $x$ and $-x$ are a pair of antipodal points on

$$\iota\left(S^m\right) \subset R^n,$$

then:

$$\begin{aligned}
d(f(x),f(-x)) &\leq d(f(x),g(x)) + d(g(x),g(-x)) + d(g(-x),f(-x)) \\
&\leq \left(\frac{3}{2}\lambda\sqrt{n}+2k\right) + d(g(x),g(-x)) + \left(\frac{3}{2}\lambda\sqrt{n}+2k\right) \\
&= d(g(x),g(-x)) + \left(3\lambda\sqrt{n}-4k\right)
\end{aligned}$$

And this gave us:

$$\begin{aligned}
d(g(x),g(-x)) &\geq d(f(x),f(-x)) - 3\lambda\sqrt{n} - 4K \\
&\geq \frac{2R}{\lambda} - 3\lambda\sqrt{n} - 5K
\end{aligned}$$

So, if we take $R > \frac{\lambda}{2}(3\lambda\sqrt{n}+5K)$, the right-hand side is positive for any pair of antipodal points, so $g(x) \neq g(-x)$, that contradicts the Borsulk-Ulam theorem.

**Example 3.1.9.** Let $G$ be a group and $S, T$ two finite generating sets, then the Cayley Graphs $\Gamma_{G,S}$ and $\Gamma_{G,T}$ are quasi-isometric under the induced word metric through the identity map $\iota : (G, d_T) \longrightarrow (G, d_S)$.

First of all, let us remember that the word metric is given by the minimum amount of letters of the generating set that represent $g \in G$ i.e., $\mathrm{d}_S(1, g)$.

As $S$ is finite let us consider

$$M_1 \coloneqq \max\{\mathrm{d}_T(e,s) \colon s \in S\},$$

which clearly is finite. Let $g, h \in G$ with $n :- d_S(g,h)$, we can write $g^{-1}h = s_1 \dots s_n$ with $s_i \in S$. So:

$$\begin{aligned}
d_T(g,h) &= d_T(g, g\cdot s_1 \cdots s_n) \\
&\leq d_T(g, g\cdot s_1) + d_T(g\cdot s_1, g\cdot s_1\cdot s_2) + \cdots + d_T(g\cdot s_1\cdots s_{n-1}, g\cdot s_1\cdots s_n) \\
&= d_T(1,s_1) + d_T(1,s_2) + \cdots + d_T(1,s_n) \\
&\leq M_1 \cdot n \\
&= M_1 \cdot d_S(g,h).
\end{aligned}$$

Interchanging the roles of $S$ and $T$ we obtain that $d_S(g,h) \leq M_2 d_T(g,h)$, for $M_2 \coloneqq \max\{\mathrm{d}_S(e,t) \colon t \in T\}$. Using $M = \max(M_1, M_2)$ we have:

$$\frac{1}{M} d_S(g,h) \leq d_T(g,h) \leq M d_S(g,h),$$

and clearly the identity has quasi-dense image, finishing the proof.

The last example allow us to give the following definition:

**Definition 3.1.10.** Let $G$ and $H$ be finitely generated groups. We say that $G \sim_{\mathrm{QI}} H$ if there exist generating sets $S_G \subset G$ and $S_H \subset H$ such that $\Gamma_{G,S_G} \sim_{\mathrm{QI}} \Gamma_{H,S_H}$.

Also if a metric space $X$ is quasi-isometric to $\Gamma_{G,S}$ for a group $G$ and a finite generating set $S$, we say that $X \sim_{\mathrm{QI}} G$.

## 3.2 The Švarc-Milnor Lemma

One natural question we can ask is, why should we be interested in understanding finitely generated groups up to quasi-isometry? The Švarc-Milnor lemma, in rough words say that with some special conditions, a group is finitely generated and is quasi-isometric to a metric space. According to Löh [13] "In practice, this result can be applied both ways, if we want to know more about the geometry of a group or if we want to know that a given group is finitely generated, it suffices to exhibit a nice action of this group on a suitable space. Conversely if we want to know more about a metric space, it suffices to find a nice action of a suitable well-known group. Therefore the Švarc-Milnor lemma is also called the fundamental lemma of geometric group theory."

Before proving the Švarc-Milnor lemma, we have to give some notions.

**Definition 3.2.1.** Let $(X, d)$ a metric space and $0 \leq L \in \mathbb{R}$. A geodesic of length $L$ in $X$ is an isometric embedding $\gamma\colon [0, L] \to X$ where the interval $[0, L]$ carries the metric induced from $\mathbb{R}$, this can be thought as a curve that in some sense is the shortest path between the start and the end. Also $\gamma(0)$ is the start point of $\gamma$ and $\gamma(L)$ is the end point of $\gamma$. The metric space $X$ is called geodesic, if for all $x, x' \in X$, there exist a geodesic in $X$ with start point $x$ and end point $x'$.

For a non-example of this, consider $X = \mathbb{R}^2 \backslash \{0\}$ with the metric induced from the euclidean metric on $\mathbb{R}^2$, note that if we take $x = (1, 0)$ and $x' = (-1, 0)$, the only possible geodesic path is the straight line, but as $\{0\} \notin X$, there is no geodesic between $x$ and $x'$.

**Definition 3.2.2.** Let $(X, d)$ be a metric space and let $0 < c$ and $0 \leq b$, a $(c, b)$-quasi-geodesic in $X$ is a $(c, b)$-quasi-isometric embedding $\gamma : I \longrightarrow X$ where $I = [t, t'] \subset \mathbb{R}$ is some closed interval. The space $X$ is $(c, b)$-quasi-geodesic if for all $x, x' \in X$, there exist a $(c, b)$-quasi-geodesic in $X$ with start point $x$ and end point $x'$.

Clearly any geodesic space is also a quasi-geodesic space, but not the other way around. For any $\varepsilon > 0 \in \mathbb{R}$, the space $X = \mathbb{R}^2 \backslash \{0\}$ is a $(1, \varepsilon)$-quasi-geodesic space as the figure 3.1 shows.

**Example 3.2.3.** If $X = (V, E)$ is a connected graph, then the associated metric on $V$ turns $V$ into a $(1, 1)$-quasi-geodesic space, because the distance between twp vertices is the length of some path in the graph.

Proposition A.3.4 from [13] shows that any quasi-geodesic space is quasi-isometric to a geodesic space.

We now come to the Švarc-Milnor lemma. First we are going to make a formulation using the language of quasi-geometries, and then we are going to deduce the "topological" version, that is the version usually used.

**Theorem 3.2.4.** *Let $G$ be a group, and let $G$ act on a metric space $(X, d)$ by isometries. Suppose that there are constants $c, b > 0 \in \mathbb{R}$ such that $X$ is a $(c, b)$-quasi-geodesic and suppose that there is a subset $B \subset X$ with the following properties:*

- *The diameter of $B$ is finite.*
- *$G$ translates $B$ over all of $X$, i.e $\bigcup_{g \in G} g \cdot B = X$.*

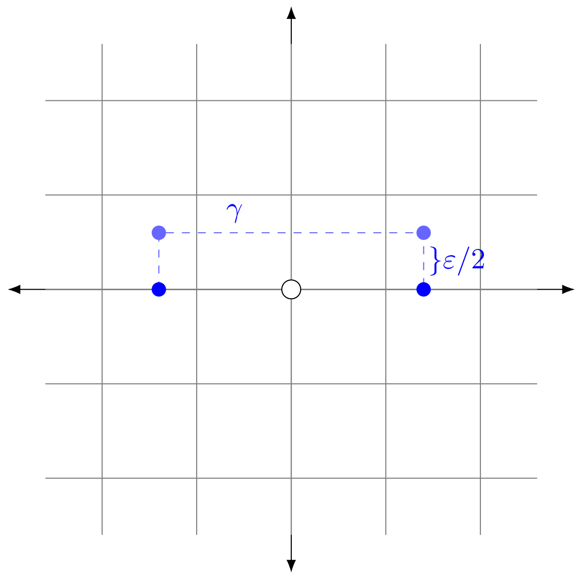



Figure 3.1: A $(1,\varepsilon)$-quasi-geodesic in $\mathbb{R}\backslash\{(0,0)\}$

- *The set* $S := \{g \in G | g \cdot B' \cap B' \neq \varnothing\} < \infty$, *where*

$$B' := \mathcal{B}_{2\cdot b}(B) = \{x \in X \colon \exists y \in B, \quad d(x,y) \leq 2 \cdot b\}.$$

*Then:*

1. *The group* $G$ *is generated by* $S$*, in particular,* $G$ *is finitely generated.*
2. *For all* $x \in X$ *the map*

$$G \longrightarrow X,$$
$$g \longmapsto g \cdot x,$$

*is a quasi-isometry with respect to the word metric on* $G$*.*

*Proof.* 1. Let $x \in B$, as $X$ is $(c,b)$-quasi-geodesic, there is a $(c,b)$-quasi-geodesic $\gamma$ of length $L$ starting in $x$ and ending in $g \cdot x$, we will define some points in this quasi-geodesic.

Let $n = \lceil L \cdot b/c \rceil$. For $j \in \{0, \ldots n-1\}$ we define:

$$t_j := j \cdot \frac{b}{c}$$

and $t_n := L$, as well,

$$x_j := \gamma(t_j).$$

Notice that $x_0 = \gamma(0) = x$ and $x_n = \gamma(L) = g \cdot x$. We know that as $G$ translates $B$ over all $X$, there are some elements $g_j \in G$ with $x_j \in g_j \cdot B$. In particular we can choose $g_0 = e \in G$ and $g_n = g$, the proceedure can be see in Figure 3.2.

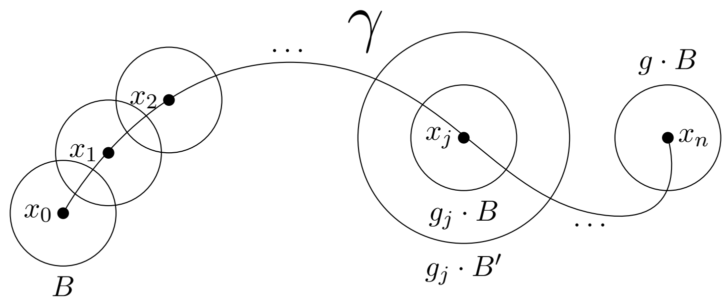



Figure 3.2: Covering a quasi-geodesic by translates of $B$

We want to show that $s_j := g_{j-1}^{-1} \cdot g_j \in S$ for $j \in \{1, \dots n\}$. For this, notice that, because $\gamma$ is a $(c,d)$-quasi-geodesic, then:

$$d(x_{j-1}, x_j) \leq c \cdot |t_{j-1} - t_j| + b \leq c \cdot \frac{b}{c} + b \leq 2 \cdot b,$$

then, $x_j \in \mathcal{B}_{2 \cdot b}(g_{j-1} \cdot B) = g_{j-1} \cdot \mathcal{B}_{2 \cdot b}(B) = g_{j-1} \cdot B'$, this is because $G$ acts on $X$ by isometries. And on the other hand, $x_j \in g_j \cdot B \subset g_j \cdot B'$, thus:

$$g_{j-1} \cdot B' \cap g_j \cdot B' \neq \varnothing.$$

So, by definition on $S$ it follows that $s_j \in S$, in particular:

$$g = g_n = g_{n-1} \cdot g_{n-1}^{-1} \cdot g_n = \cdots = g_0 \cdot s_1 \cdot \cdots \cdot s_n = s_1 \cdot \cdots \cdot s_n$$

lies in the set generated by $S$, as desired. We did this for any $g \in G$, so $S$ generates $G$.

2. We will show that the map

$$\begin{aligned} \varphi : & G \longrightarrow X \\ & g \longmapsto g \cdot x \end{aligned}$$

is a quasi-isometry by showing that it is a quasi-isometric embedding with quasi-dense image. Let $x \in X$, we may assume that $B$ contains $x$ because $G$ translates B over all $X$. Let us consider another $x' \in X$, then there is a $g \in G$ with $x' \in g \cdot B$, also $g \cdot x \in g \cdot B$, then

$$d\left(x', \varphi(g)\right) = d\left(x', g \cdot x\right) \leq \operatorname{diam} g \cdot B = \operatorname{diam} B,$$

thus $\varphi$ has quasi-dense image.

Now it only remains to show that $\varphi$ is a quasi-isometric embedding. First we give a lower bound of $d(\varphi(e), \varphi(g)$ in terms of the distance given by the set $S$, $d_S(e,g)$. As above let $\gamma : [0, L] \longrightarrow X$ to be a $(c,b)$-quasi-geodesic from $x$ to $g \cdot x$, then the first part and the definition of $n$ shows that:

$$
\begin{aligned}
d(\varphi(e), \varphi(g)) = d(x, g \cdot x) = d(\gamma(0), \gamma(L)) \\
&\geq \frac{1}{c} \cdot L - b \\
&\geq \frac{1}{c} \cdot \frac{b \cdot (n-1)}{c} - b \\
&= \frac{b}{c^2} \cdot n - \frac{1}{c^2} - b \\
&\geq \frac{b}{c^2} \cdot d_S(e, g) - \frac{1}{c^2} - b.
\end{aligned}
$$

Now we want to give an upper bound of $d(\varphi(e), \varphi(g))$ in terms of $d_S(e, g)$. Even if the coefficients are different from the lower bound, using the max of both bounds we get the inequality we are looking for.

Suppose $d_S(e, g) = n$, there are $s_1 \ldots s_n$, with $s_i \in S$ such that $g = s_1 \cdots s_n$. As $s_j \cdot B' \cap B' \neq \varnothing$, and the $x_i$ are located on the geodesic, if we take $y \in B' \cap s \cdot B'$, there exists some $x_1 \in B' : d(x_1, y) < 2b$ and $s \cdot x_2 \in s \cdot B' : d(x_2, y) < 2b$, therefore

$$
\begin{aligned}
d(x, s \cdot x) &\leq d(x, x_1) + d(x_1, y) + d(y, s \cdot x_2) + d(s \cdot x_2, s \cdot x) \\
&\leq \operatorname{diam} B + 2b + 2b + \operatorname{diam} B \\
&\leq 2(\operatorname{diam} B + 2b),
\end{aligned}
$$

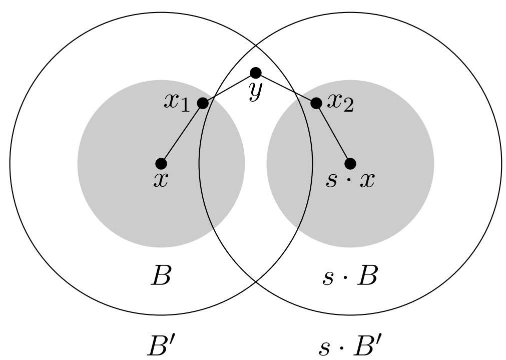



Figure 3.3: $B \cap s$·B

as figure 3.3 shows.
Using that G acts isometrically on $X$, we obtain:

$$
\begin{aligned}
d(\varphi(e), \varphi(g)) &= d(x, g \cdot x) \\
&\leq d(x, s_1 \cdot x) + d(s_1 \cdot x, s_1 \cdot s_2 \cdot x) + \cdots + d(s_1 \cdot \cdots \cdot s_{n-1} \cdot x, s_1 \cdot \cdots \cdot s_n \cdot x) \\
&= d(x, s_1 \cdot x) + d(x, s_2 \cdot x) + \cdots + d(x, s_n \cdot x) \\
&\leq n \cdot 2 \cdot (\operatorname{diam} B + 2 \cdot b) \\
&= 2 \cdot (\operatorname{diam} B + 2 \cdot b) \cdot d_S(e, g)
\end{aligned}
$$

Recall that $\operatorname{diam} B$ is assumed to be finite, and because:

$$
d(\varphi(g), \varphi(h)) = d\left(\varphi(e), \varphi\left(g^{-1} \cdot h\right)\right)
$$

and $d_S(g,h) = d_S\left(e, g^{-1}\cdot h\right)$, the bounds show that $\varphi$ is an quasi-isometric embedding.

□

Before deducing the "Topological version" of the Švarc-Milnor lemma, we briefly recall some topological notions.

**Definition 3.2.5.** A metric space $X$ is proper if for all $x \in X$ and all $0 \leq r \in \mathbb{R}$, the closed ball with center $x$ and radius $r$ is compact with respect to the topology induced by the metric. Notice that proper metric spaces are locally compact.

**Definition 3.2.6.** An action $G \times X \longrightarrow X$ of a group $G$ on a topological metric space $X$, is proper if for all compact sets $B \subset X$ the set $\{g \in G \colon g \cdot B \cap B \neq \varnothing\}$ is finite.

**Example 3.2.7.** The translation action of $\mathbb{Z}$ on $\mathbb{R}$ is proper.

**Lemma 3.2.8.** The action by deck transformations on the fundamental group of a locally compact path-connected topological space on its universal covering is proper.

*Proof.* Because of the definition of deck transformations, for each $y \in Y$ ($Y$ universal covering space of $X$), there exists $U_y$ neighborhoods, such that if $U_y \cap g \cdot U_y \neq \varnothing$, then $g = e$. (See A.2.6.)

Consider the following lemma, for any compact $B \in Y$, $(g \cdot U_y) \cap B \neq \varnothing$ for only finite $g$, with $y \in B$. To prove it let us define $C := \{g \in G \colon (g \cdot U_y) \cap B \neq \varnothing\}$, for any $g \in C$ there exists $x_g \in U_y$ such that $g \cdot x_g \in B$, (i.e $x_g \in g \cdot U_y \cap B$). Consider the following function:

$$\begin{aligned} \varphi : &C \longrightarrow \bigcup_{g \in C} (g \cdot U_y) \cap B \\ &g \longmapsto g \cdot x_g. \end{aligned}$$

Notice that $\varphi$ is injective. Now, $g \cdot x_g \in g \cdot U_y$, with $U_y$ neighborhood such that if $g \neq g'$, then $g \cdot U_y \cap g' U_y = \varnothing$, therefore $\{g \cdot x_g \colon g \in C\}$ is a discrete subset in $B$, and as $B$ is compact, $C$ is finite.

Clearly

$$g \cdot B \cap B \subseteq \left( \bigcup_{i=1}^{n} g \cdot U_{y_i} \right) \cap B,$$

but because of the lemma,

$$\bigcup_{g \in G} \bigcup_{i=1}^{n} g \cdot U_{y_i} \cap B,$$

only has finite terms. □

**Definition 3.2.9.** An action $G \times X \longrightarrow X$ of a group $G$ on a topological space $X$ is cocompact if the quotient space $G \backslash X$ with respect to the quotient topology is compact.

**Example 3.2.10.**
- The translation action of $\mathbb{Z} \curvearrowright \mathbb{R}$ is cocompact, because the quotient is homeomorphic to the circle $S^1$.

- The horizontal translation of $\mathbb{Z} \curvearrowright \mathbb{R}^2$ is not cocompact because the quotient is homeomorphic to the infinite cylinder $S^1 \times \mathbb{R}$.
- The action by deck transformations of the fundamental group of a compact path-connected topological space $X$ on its universal covering is cocompact because the quotient is homeomorphic to $X$

**Lemma 3.2.11.** Let $X$ be a space and let $G$ be a group which acts by homeomorphisms on $X$. Then the map $\pi : X \longrightarrow X/G$ is open.

*Proof.* Let $U \subset X$ an open set, then its image $\pi(U)$ is open if and only if $\pi^{-1}(\pi(U))$ is open by definition of the quotient topology. Also, $\pi^{-1}(\pi(U)) \neq U$, but

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U,$$

as $G$ acts as a homeomorphism $g \cdot U$ is an open set for any $g$. □

With this in mind we can formulate the Švarc-Milnor lemma in its topological version.

**Corollary 3.2.12.** Let $G$ be a group acting by isometries on a proper geodesic metric space $(X, d)$, furthermore, suppose that this action is proper and cocompact. Then $G$ is finitely generated and for all $x \in X$ the map

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto g \cdot x \end{aligned}$$

is a quasi-isometry.

*Proof.* First of all, notice that under the given assumptions, $X$ is a $(1, \varepsilon)$-quasi-geodesic space for all $\varepsilon \geq 0$. If we want to apply the Theorem 3.2.4, we need to find a nice subset $B \subset X$.

Because of Lemma 3.2.11, the natural projection $\pi : X \longrightarrow G \backslash X$ is an open map, on the other hand $G \backslash X$ is compact, so one can find a closed subspace $B \subset X$ of finite diameter (for example a suitable union of finitely many closed balls), such that $\pi(B) = G \backslash X$, where $\pi$ is the projection $\pi : X \longrightarrow G \backslash X$ associated with the action of $G$. In particular

$$\bigcup_{g \in G} g \cdot B = X$$

and $B' := B_{2 \cdot \varepsilon}(B)$ has finite diameter. Because $X$ is proper, the subset $B'$ is compact, thus the action of $G$ on $X$ being proper implies that $\{g \in G | g \cdot B' \cap B' \neq \varnothing\}$ is finite, hence we can apply Theorem 3.2.4 □

**Corollary 3.2.13.** Let $M$ be a compact and without boundary connected Riemannian manifold, and $\widetilde{M}$ be its Riemannian universal covering manifold. Then the fundamental group $\pi_1(M)$ is finitely generated and for every $x \in \widetilde{M}$, the map

$$\begin{aligned} \pi_1(M) &\longrightarrow \widetilde{M} \\ g &\longmapsto g \cdot x \end{aligned}$$

given by the action of the fundamental group on $\widetilde{M}$ via deck transformations is a quasi-isometry.

*Proof.* If we want to apply the topological version of the Švarc-Milnor lemma, we have to use some things. First of all notice that $\pi_1(M)$ acts by isometries in $\widetilde{M}$ via deck transformations (A.2.5). We need to show that $\widetilde{M}$ is a proper geodesic metric space. The Proposition B.2.3 gives us the metric that is induced by $M$, and the fact that $M$ is compact (this implies that is complete, and therefore $\widetilde{M}$ is also complete) gives us the conditions we need thanks to Theorem B.2.8. Clearly the action of $\pi_1(M)$ is cocompact (because $M$ is compact) and the Lemma 3.2.8 shows that the action is also proper, then the result follows. □

There are many applications of the Švarc-Milnor lemma, a basic example is the following:

**Corollary 3.2.14.** Finite index subgroups of finitely generated groups are finitely generated and quasi-isometric to the ambient group.

*Proof.* Let $G$ be a finitely generated group with generating set $S$ and $H \subset G$ be a subgroup of finite index, if we consider the action of $H$ on $G$ by left translation, notice the following facts:

- The space $(G, \mathrm{d}_S)$ is $(1,1)$-quasi-geodesic space.
- Let $B \subset G$ be a finite set of representatives of $H \backslash G$, then $B$ has finite diameter.
- $H \cdot B = G$
- The set $B'$ is finite as well, and moreover the set

$$\{h \in H \colon h \cdot B' \cap B' \neq \varnothing\}$$

  is finite.

With all these we can apply the topological version of Švarc-Milnor lemma and therefore $H$ is finitely genereated and the inclusion $H \hookrightarrow G$ is a quasi-isometry. □

This last corollary give us a motivation to the following definition

**Definition 3.2.15.**
- Two groups $G$ and $H$ are commensurable if they contain $G' \subset G$ and $H' \subset H$ finite index subgroups with $G' \cong H'$.
- Two groups $G$ and $H$ are weakly commensurable if they contain finite index subgroups $G' \subset G$ and $H' \subset H$ that satisfy that there are finite normal subgroups $N \trianglelefteq G'$ and $M \trianglelefteq H'$ such that $G'/N$ and $H'/M$ are isomorphic.

Clearly, because of Corollary 3.2.14, if $G$ and $H$ are commensurable, then they are quasi-isometric.

**Corollary 3.2.16.** Let $G$ be a group, then:

1. Let $G'$ be a finite index subgroup of $G$, then $G'$ is finitely generated iff $G$ is finitely generated. If these groups are finitely generated, then $G \sim_{\mathrm{QI}} G'$.
2. Let $N$ be a finite normal subgroup, then $G/N$ is finitely generated iff $G$ is finitely generated. If these groups are finitely generated, then $G \sim_{\mathrm{QI}} G/N$.

In particular, if $G$ is finitely generated, then any group weakly commensurable to $G$ is finitely generated and quasi-isometric to $G$.

*Proof.* 1. With Corollary 3.2.14, it is enough to show that $G$ is finitely generated if $G'$ is. If we combine the finite generating set of $G'$ and a finite set of representatives of the $G'$-cosets in $G$, this yields a finite generating set of $G$.

2. 
   - If $G$ is finitely generated, then $G/N$ is finitely generated.
   - Conversely if $G/N$ is finitely generated, similar to the item 1, combining the finite set $N$ with the lifts with respect to the canonical projection of $G \to G/N$ of a generating sets of $G/N$ gives a finite generating set of $G$.
   - Let $G$ and $G/N$ finitely generated with $S$ a generating set of $G/N$, again similar to Corollary 3.2.14 we take $B$ as $G/N$, that is finite and the left-translate action of $G$ over $G/N$ translates $B$ all over $G$, therefore applying Švarc-Milnor (3.2.4) we have the quasi-isometry.

□

In particular, if $G$ is finitely generated, then any group weakly commensurable to $G$ is finitely generated and quasi-isometric to $G$.

**Example 3.2.17.** It is known that for $n \geq 2$ then the free group of rank 2 contains a free group of rank $n$ as a finite index subgroup, i.e $\mathbb{F}_n \leqslant \mathbb{F}_2$. Also $\mathbb{F}_n \leqslant \mathbb{F}_n$, therefore they are commensurable. This shows that all free groups of finite rank bigger than 1 are quasi-isometric.

## 3.3 Quasi-isometry invariants

The Švarc-Milnor lemma is in some way an attempt to classify finitely generated groups up to quasi-isometry. One important problem is to construct properties on groups preserved up to quasi-isometries. A common name for this quasi-isometric invariants, are geometric properties. A similar concept was introduced in Chapter 2 with the large-scale properties that does not depend on the generating set. Example 3.1.9 shows that quasi-isometries does not depend on the generating set, so the geometric properties are more general than large-scale properties.

A simple case of a quasi-isometric invariant is the finiteness, as the Example 3.1.4 shows. From a coarse geometric point of view, if we draw the Cayley graph of a finite group and go far away enough, we are eventually looking at a dot. Clearly, because of that, being abelian is not a geometric property of groups.

Example 3.2.17 shows that the rank of free groups is not a quasi-isometric invariant.

**Remark 3.3.1.** For any property of groups, a group has virtually that property if it contains a finite index subgroup that has that property.

There are many other properties that are geometric, they can be consulted in Example 5.5.11, page 143 of Clara Löh's book [13], for example:

- Being virtually $\mathbb{Z}^n$.
- Being virtually nilpotent.

- Being finitely presented.

**Theorem 3.3.2.** *Let $G$ and $H$ finitely generated groups, and let $S \subset G$ and $T \subset H$, finite generating sets. (Using the notation of the Definition 2.3.9)*

1. *If there exist a quasi-isometric embedding $(G, d_S) \to (H, d_T)$, then*

$$\beta_{G,S} \preceq \beta_{H,d_T}.$$

2. *In particular, if $G$ and $H$ are quasi isometric, then the growth functions are equivalent.*

*Proof.* Let $f : G \to H$ be a quasi-isometric embedding, there is a $c \in \mathbb{R}_{>0}$ such that

$$\forall g, g' \in G \quad \frac{1}{c} \cdot d_S\left(g, g'\right) - c \leq d_T\left(f(g), f\left(g'\right)\right) \leq c \cdot d_S\left(g, g'\right) + c$$

We name $x = f(e_G)$, and let $r \in \mathbb{N}$. Note the following:

- If $g \in \mathcal{B}_S(e_G, r)$, then $d_T(f(g), x) \leq c \cdot d_S(g, e_G) + c = c \cdot r + c$, therefore

$$f\left(\mathcal{B}_S(e_G, r)\right) \subset \mathcal{B}_T(x, c \cdot r + c).$$

- If $g_1, g_2 \in G$ such that $f(g_1) = f(g_2)$,

$$d_S(g_1, g_2) \leq c \cdot (d_T(f(g_1), f(g_2) + c) = c^2.$$

These two facts give us the following inequality, the first line holds because we can use the two inequalities from above for any $g \in \mathcal{B}_S(e_G, r)$, and the second because of the definition of $d_S$ and $d_T$, those metrics are invariant under left translation, so:

$$\begin{aligned}\beta_S(r) &\leq \left|\mathcal{B}_S(e_G, c^2)\right| \cdot \left|\mathcal{B}_T(x, c \cdot r + c)\right| \\ &= \left|\mathcal{B}_S(e, c^2)\right| \cdot \left|\mathcal{B}_T(e_H, c \cdot r + c)\right| \\ &= \beta_S(c^2) \cdot \beta_T(c \cdot r + c).\end{aligned}$$

Notice that the term $\beta_S(c^2)$ does not depend on the radius, so the last inequality shows that $\beta_{G,S} \preceq \beta_{H,d_T}$. The second item follows from the first one. □

This theorem shows that the growth type is a geometric property.

**Definition 3.3.3.** Let $\Gamma$ be a connected, locally finite graph (for example a Cayley's graph of a finitely generated group), and let $\mathcal{B}(n)$ be the ball of radius $n$ in $\Gamma$, based at some fixed vertex, we define $||\Gamma \backslash \mathcal{B}(n)||$ to be the number of connected unbounded components of the complement of $\mathcal{B}(n)$.

**Lemma 3.3.4.** Let $\Gamma$ be a connected, locally finite graph and $m \leq n$ two positive integers, then

$$||\Gamma \backslash \mathcal{B}(m)|| \leq ||\Gamma \backslash \mathcal{B}(n)||.$$

*Proof.* Let $\mathcal{C}$ be an unbounded connected component of $\Gamma \backslash \mathcal{B}(m)$, either $\mathcal{C}$ remains connected when we remove $\mathcal{B}(n)$. □

**Definition 3.3.5.** Let $\Gamma$ be a connected, locally finite graph, and let $\mathcal{B}(n)$ be the ball of radius $n$ in $\Gamma$, based at some fixed vertex $v \in V(\Gamma)$, we define the number of ends of $\Gamma$ as:

$$e(\Gamma) = \lim_{n \to \infty} ||\Gamma \backslash \mathcal{B}(n)|| \, .$$

Since by the Lemma 3.3.4 the sequence $\{||\Gamma \backslash \mathcal{B}(n)||\}_{n \in \mathbb{N}}$ is a non-decreasing sequence of integers, the limit exists.

**Example 3.3.6.** Consider the Cayley Graph of $\mathbb{Z}$ with $S = \{1\}$, note that for any $n \in \mathbb{N}$, if we consider $\Gamma_{\mathbb{Z},\{1\}} \backslash \mathcal{B}(n)$, there always are two connected components, the positive one and the negative one, as in Figure 3.4.

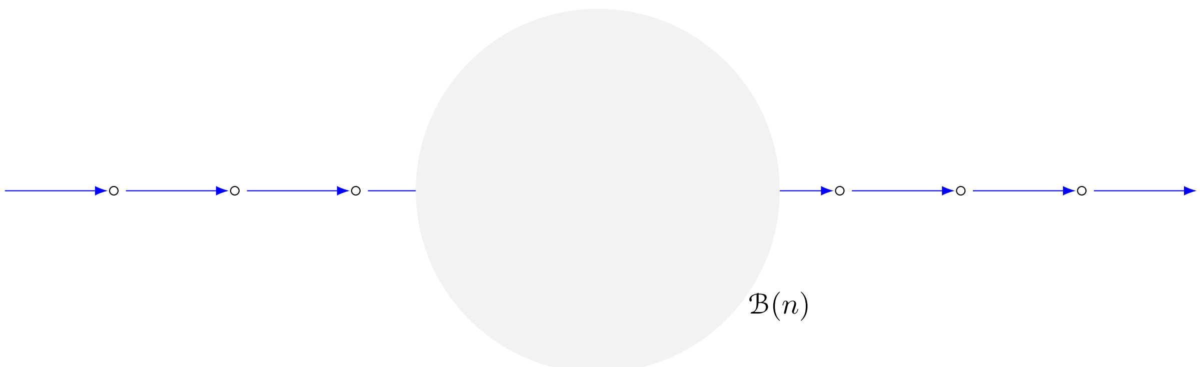



Figure 3.4: $|\ \Gamma_{\mathbb{Z},\{1\}} \backslash \mathcal{B}(n) \|$

**Example 3.3.7.** It is easy to see that for $\mathbb{Z}^2$, $e(\Gamma_{\mathbb{Z}^2,S}) = 1$ for $S$ a finite generating set.

**Example 3.3.8.** Note that if we have a finite group $G$, for any finite generating set S, we have that $e(\Gamma_{\mathbb{G},S}) = 0$.

**Remark 3.3.9.** There are several different definitions of the ends of a space, this concept can be extended to any metric space, but anyway according to Clara Löh [13], all these definitions are equivalent when they are applied to groups.

The last example gives an intuitive notion of the idea that the ends of a group is a geometric property, in fact proposition 8.2.5 of [13] shows that if two spaces are quasi-isometric, in particular the have the same amount of ends.

There are other important geometric properties of groups such as hyperbolicity or amenability that need more theory to prove that they are quasi-isometric invariants, but these notations are very important to the study of geometric group theory, for more information this can be found also in [13].

# Appendix A

# Algebraic Topology

## A.1 Fundamental Group

Before defining the fundamental group, we need some other notions.

**Definition A.1.1.** A path in an space (not necessarily metric) $X$ is a continuous map $f : I \to X$, where $I = [0, 1] \in \mathbb{R}$.

The idea of the fundamental group is consider the deformation of the paths keeping the endpoints, this is formally the next definition.

**Definition A.1.2.** A homotopy of paths in $X$ is a family $f_t : I \to X$, with $0 \leq t \leq 1$ such that:

- $f_t(0) = x_0$ and $f_t(1) = x_1$ for any $t$.
- The map $F : I \times I \to X$ defined by $F(s, t) = f_t(s)$ is continuous.

When two paths $f$ and $g$ are connected by an homotopy, they said to be homotopic and we write it as $f \simeq g$.

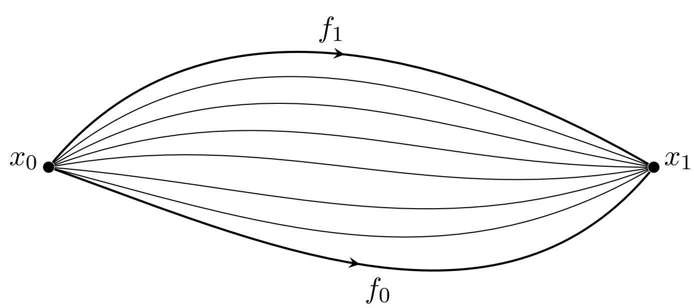



Figure A.1: Homotopy of paths

**Proposition A.1.3.** The relation of homotopy of paths with fixed endpoints in a space $X$ is an equivalent relation.

*Proof.*

- Reflexivity: Defining $f_t = f$, we have that $f \simeq f$.
- Symmetry: Given an homotopy of paths $f_t$ of $f_0$ and $f_1$, the homotopy $f_{1-t}$ gives another between $f_1$ and $f_0$.
- Transitivity: If $f_0 \simeq f_1$ via $f_t$ and $f_1 \simeq g_0$ via $g_t$ we can consider the homotopy $h_t$ defined as:

$$h_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}$$

  Notice that there is no problem in $t = \frac{1}{2}$ since we are assuming $f_1 = g_0$. We have the continuity if $H(s,t) = h_t(s)$ because a function defined on the union of two closed sets is continuous if it is continuous when restricted to each of the closed sets separately. Since $H$ is continuous on $I \times \left[0, \frac{1}{2}\right]$ (because of $F$) and on $I \times \left[\frac{1}{2}, 1\right]$ (because of $G$), it is continuous on $I \times I$.

□

The equivalence relation of a path $f$ is called an homotopy class and is denoted as $[f]$. We can define an operation between paths. Given to paths $f, g : I \to X$ with $f_1 = g_0$ we define the composition (or concatenation) as:

$$(f \star g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

This is travel through both paths twice as fast in order to do it in unit time. Also, this operation respects the homotopy class, if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via homotopies $f_t$ and $g_t$, and $f_0(1) = g_0(1)$, then $(f_0 \star g_0) \simeq (f_1 \star g_1)$. In particular we can restrict this to paths with the same start and ending points, i.e, $f(0) = f(1) = x_0$.

The Proposition 1.3 of Hatcher's Book [9] proves the next result.

**Theorem A.1.4.** *$\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \star g]$.*

**Example A.1.5.** Let $X = S^2$, note that as it is simply connected, any loop can is contractable to a single point. So for any $x \in S^2$, $\pi_1(S^2, x) = 0$.

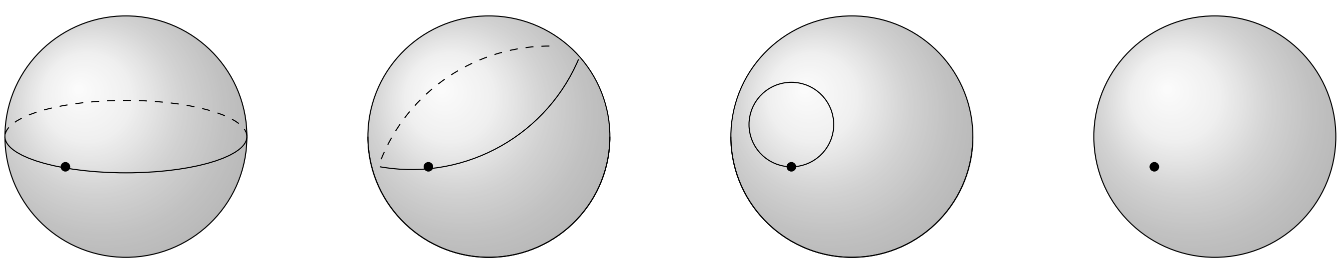


Figure A.2: Loops in $S^2$

**Example A.1.6.** Another example is $X = S^1$, whose fundamental group is $\pi_1(S^1) \cong \mathbb{Z}$, this intuitively can be see that any closed curve in $S^1$ is on the number of loops the curve does through the circle.

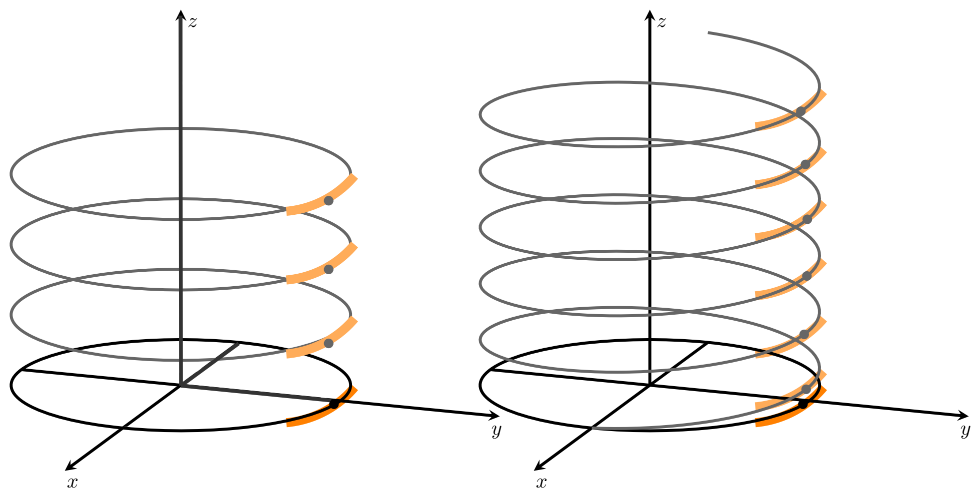
There are a lot of tools that can be used to calculate the fundamental group of a topological space, the interested reader can find them in [9] or [12].

## A.2 Covering Spaces

One of the tools that are used to calculate fundamental groups and that are related with those, are the covering spaces.

**Definition A.2.1.** A covering space of a space $X$ is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ that satisfies that there exists an open cover $\{U_\alpha\}$ of $X$ such that for any $\alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto $U_\alpha$ by $p$.

**Example A.2.2.** Consider $X = S^1$ in the $(x, y)-$plane, the helix parameterized by $(\cos 2\pi t, \sin 2\pi t, t)$ is a covering space with $\mathcal{P}$ the natural projection on the $(x, y)-$plane, as in the figure A.3.




(a) Covering space of $S^1$ with circles (b) Covering space of $S^1$ with an helix

Figure A.3: Covering spaces of $S^1$

In general there can be many covering spaces of a space, for example, if we just set circles one over each other as in A.3a is another covering space, even if it is not connected. This allow us to give the following definition.

**Definition A.2.3.** Let $\widetilde{X}$ be a covering space, if it is simply connected.

**Theorem A.2.4.** *Corollary 4.6, page 143, Bredon [12]. Let* $p_i : \widetilde{X_i} \to X$*,* $i = 1, 2$*, be covering spaces that* $\widetilde{X_1}$ *and* $\widetilde{X_2}$ *are both simply connected. If* $\tilde{x}_i \in \widetilde{X_i}$*are such that* $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$*, then there is a unique map* $g : \widetilde{X_1} \to \widetilde{X_2}$ *such that* $p_2 \circ g = p_1$ *and* $g(\tilde{x}_1) = \tilde{x}_2$*. Moreover,* $g$ *is a homeomorphism.*

This theorem is the reason why the simply connected covering spaces are named *universal* covers.

There are several facts about covering spaces that are related with the fundamental group, that also can be found in [9] or [12].

**Definition A.2.5.** Let $p : \widetilde{X} \to X$ a covering map the isomorphisms $\widetilde{X} \to \widetilde{X}$ are called deck transformations.

These form a group under composition ($G(\widetilde{X})$). For example for example A.2.2 the deck transformations are the vertical translation of the helix onto itself, so $G(\widetilde{X}) \cong \mathbb{Z}$.

One fact that we used on 3.2.8 is the following lemma:

**Proposition A.2.6.** The action of deck transformation group satisfies that for each $\tilde{x} \in \widetilde{X}$ has a neighboor $U$ such that for any $g_1, g_2 \in G(\widetilde{X})$, $g_1 \cdot U \cap g_2 \cdot U \neq \varnothing$ implies $g_1 = g_2$.

*Proof.* Let $\widetilde{U} \subset \widetilde{X}$ projects (via $p$) to $U \subset X$. If $g_1 \cdot \widetilde{U} \cap g_2 \cdot \widetilde{U} \neq \varnothing$ for some $g_1, g_2 \in G(\widetilde{X})$, then $g_1(\tilde{x}_1) = g_2(\tilde{x}_2)$ for some $\tilde{x}_1, \tilde{x_2} \in \widetilde{U}$. Note that $\tilde{x}_1$ and $\tilde{x_2}$ must lie in the same fiber $p^{-1}(x)$, which intersects $\widetilde{U}$ in only one point, then $\tilde{x}_1 = \tilde{x}_2$, therefore $g_1^{-1} g_2$ fixes $x$, so $g_1^{-1} g_2(U) = \text{id}$, so $g_1 = g_2$. □

Note also that this condition is equivalent to the condition that $U \cap g \cdot U \neq$ only when $g$ is the identity.

# Appendix B

# Riemannian Geometry

## B.1 Differential Manifolds

**Definition B.1.1.** Let $(M, \mathfrak{T})$ be a topological Hausdorff space with countable basis, we say that $M$ is a topological manifold if there exists an $m \in \mathbb{Z}^+$ such that for any point $p \in M$ we have an open neighborhood $U$ of $p$, an open $V \subset \mathbb{R}^m$ and an homeomorphism $\phi : U \to V$.

The pair $(U, \phi)$ is called a local chart (parameterization or system of coordinates) on $M$ and the integer $m$ is called the dimension of $M$.

Also, if $M$ is a $m-$dimensional topological manifold, then a $C^r-$atlas on $M$ is a collection:

$$\mathfrak{U} = \{(U_\alpha, \phi_\alpha) \colon \alpha \in \mathfrak{I}\}$$

of local charts such that $\mathfrak{U}$ covers $M$, i.e $M = \bigcup_\alpha U_\alpha$ and for any $\alpha, \beta \in \mathfrak{I}$ the transition maps i.e $\phi_\beta \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi_\beta^{-1}$ are $r-$times continuously differentiable.
An atlas $\mathfrak{M}$ is said to be maximal (or differentiable structure) if it is not contained in a larger atlas, i.e if $\mathfrak{U}$ is any other atlas containing $\mathfrak{M}$ then $\mathfrak{U} = \mathfrak{M}$.

**Definition B.1.2.** An atlas $\{U_\alpha, \phi_\alpha\}$ is called differentiable if all chart transitions

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are differentiable of class $C^\infty$.

**Definition B.1.3.** A differentiable manifold of dimension $d$ is a topological manifold of dimension $d$ with a differentiable structure.

**Definition B.1.4.** A map $g : M \to M'$ between differentiable manifolds $M$ and $M'$ with charts $\{U_\alpha, \phi_\alpha\}$ and $\{U'_\alpha, \phi'_\alpha\}$ is called differentiable if all maps $\phi'_\beta \circ g \circ \phi_\alpha^{-1}$ are differentiable where defined.

The set of tangent vectors to a differential manifold $M$ in a point $p$ have a natural structure of real vector space, and we denote it as $T_pM$, an example is shown in figure B.1.

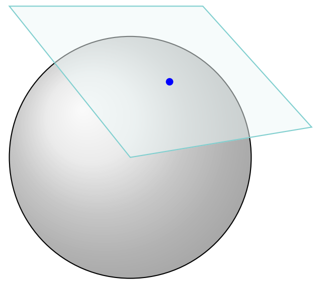


Figure B.1: Tangent space on $S^2$

**Definition B.1.5.** We define the tangent bundle of $M$ as the disjoint union of the tangent spaces to $M$ for all points in $M$, i.e:

$$TM = \bigcup_{p\in M} T_pM = \bigcup_{p\in M} \{p\} \times T_pM$$

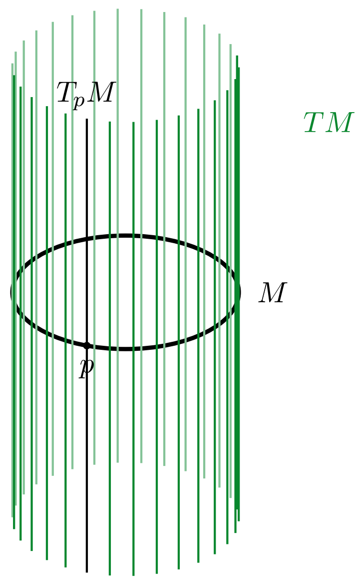



Figure B.2: Tangent Bundle of $S^1$

**Example B.1.6.** Consider the figure B.2, with the manifold $M = S^1$ The tangent bundle of the circle is isomorphic to $S^1 \times \mathbb{R}$. Geometrically, this is a cylinder of infinite height.

## B.2 Riemannian Manifolds

**Definition B.2.1.** A (smooth) vector field on a manifold $M$ is a smooth is a smooth map $X$ from $M$ to $TM$ such that for any $p \in M$, $X(p) \in T_pM$.

**Definition B.2.2.** Let $M$ be a differentiable manifold of dimension $n$, a Riemannian metric on $M$ is a family of inner products

$$g_p : T_pM \times T_pM \to \mathbb{R},$$

for $p \in M$, such that for every pair of vector fields $X$, $Y$ on $M$, the map $p \mapsto g_p(X|_p, Y|_p)$ defines a smooth function from $M$ to $\mathbb{R}$.

The pair $(M, g)$ is called a Riemannian manifold that with the Riemannian metric is a metric space.

**Proposition B.2.3.** If $M$ is a Riemannian manifold then its universal cover $\widetilde{M}$ is also a Riemannian manifold.

*Proof.* We can equip $M$ with the Riemannian metric $d$, notice that the projection $\pi : \widetilde{M} \to M$ is a local diffeomorphism, then for any $p \in \widetilde{M}$, the differential of $\pi$ is $d\pi_p : T_p\widetilde{M} \to T_{\pi(p)}M$ wich allow us to define an inner product on $T_p\widetilde{M}$ from the inner product on $T_{\pi(p)}M$, more explicitly $\langle x, y\rangle_{\widetilde{M}}(p) = \langle d\pi(x), d\pi(y)\rangle_M(\pi(p))$, doing so for each $p \in \widetilde{M}$ defines a metric $\tilde{d}$ on $\widetilde{M}$, whose smoothness follows from smoothness of $d$ and the fact that $\pi$ is a local diffeomorphism. □

The geodesics in Riemannian manifolds are a very important in the study of Riemannian geometry, and they are a little different from the geodesic paths that we have been using in the whole document. Theorem 7.22 of [14] shows that in a Riemannian manifold the geodesics are locally the shortest path between their end points, while in our definition (3.2.1) the geodesics are globally the shortest path between two points.

**Definition B.2.4.** A parameterized curve $\gamma : I \to M$ is a geodesic at $t_0 \in I$ if $\frac{D}{dt}(\frac{d\gamma}{t}) = 0$ at $t_0$, if $\gamma$ is a geodesic at $t$ for all $t \in I$ we say that $\gamma$ is a geodesic. The restriction of $\gamma$ to $[a, b]$ is called a geodesic segment joining $\gamma(a)$ to $\gamma(b)$.

Do Carmo [15] in the third chapter (page 64) gives the following fact:
Lets consider $(U, \phi)$ a chart at $p \in M$ ($M$ a is a n-dimensional manifold), there exists an open set $\mathcal{U} \in TU$, such that $(p, 0) \in \mathcal{U}$ (Notice that $TU \simeq U \times \mathbb{R}^n$), a number $\delta > 0$ and a $C^\infty$ mapping $\varphi : (-\delta, \delta) \times \mathcal{U} \to TU$ such that $t \mapsto \varphi(t, q, v)$ is the unique trajectory of $G$ which satisfies the initial condition $\varphi(0, q, v) = (q, v)$ for each $(q, v) \in \mathcal{U}$. It is possible to choose $\mathcal{U}$ in the form

$$u = \{(q, v) \in TU \colon q \in V \text{ and } v \in T_qM \text{ with } |v| < \varepsilon_1\},$$

where $V \subset U$ is a neighborhood of $p$.

**Theorem B.2.5.** *Let $\mathcal{U} \subset TU$ be an open set and $p \in M$, the map $\exp_p : \mathcal{U} \to M$ given by*

$$\exp(q, v) = \gamma(1, q, v) = \gamma\left(|v|, q, \frac{v}{|v|}\right), \quad (q, v) \in \mathcal{U}$$

*is called the exponential map on $\mathcal{U}$.*

**Definition B.2.6.** A Riemannian manifold $M$ is geodesically complete is for all $p \in M$, the exponential map $\exp_p$ is defined for all $v \in T_pM$

As we mentioned before the geodesics in Riemannian manifolds are curves of minimum length locally, but the concept of geodesically complete extend this to a global property.

**Definition B.2.7.** The distance $d(p, q)$ is defined by $d(p, q) =$ infimum of the lengths of all curves $f_{p,q}$ where $f_{p,q}$ is a piecewise differentiable curve joining $p$ to $q$.

If there exists a minimizing geodesic $\gamma$ joining $p$ to $q$ then $d(p,q) =$ length of $\gamma$. The existence of this minimizing geodesic is not always guaranteed, the following theorem gives the existence of those under different conditions. The proof of this can be found in theorem 2.8 of do Carmo's book [15]

**Theorem B.2.8** (Hopf-Rinow)**.** *Let $M$ be a Riemannian manifold and $p \in M$ then the following are equivalent:*

*a) $exp_p$ is defined on all $T_pM$.*

*b) The closed and bounded sets of $M$ are compact.*

*c) $M$ is complete as a metric space.*

*d) $M$ is geodesically complete.*

*e) There exits a sequence of compact subsets $K_n \subset M$, $K_n \subset K_{n+1}$ and $\bigcup_n K_n = M$, such that if $q_n \notin K_n$, then $d(p,q_n) \to \infty$.*

*Also, any of the statements above implies that:*

*f) For any $q \in M$ there exists a geodesic $\gamma$ joining $p$ to $q$ with $d(p,q) = length(\gamma)$.*

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